

DUALITY FOR SIN-GROUPS

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1. INTRODUCTION

With this paper, we hope to give a new look on the old problem of duality of locally compact and quantum groups. Dating back to 1930s, this theory is still under active development, and one might confess that in the general case it has not reached the excellence of the initial Abelian theorem of Pontryagin and van Kampen [31, 39].

Let us recall the statement of the problem. In contrast to Abelian groups, the space of irreducible representations of a noncommutative locally compact group has no natural group structure. One has either to consider duality between objects of different type, as it is done by Tannaka [36], Kreĭn [23] and Tatsuuma [37], or to imbed groups into a new category with duality so that the Abelian duality is preserved.

The second program has been realized by efforts of many authors, notably Kac, Vainerman, Woronowicz, Enock and Schwartz. A historical account may be found in [12] or [17]. In this theory, locally compact groups are imbedded into the category of Kac algebras, which are von Neumann algebras with additional Hopf-like structure. For a group G , this von Neumann algebra is $L^\infty(G)$, the algebra of bounded measurable functions on G .

Further extension of this theory to quantum groups was started by Woronowicz [40] and Baaj and Skandalis [5]. In fact, there are several theories: compact quantum groups of Woronowicz [41], discrete quantum groups of Effros and Ruan [10] and Van Daele [38], and two different definitions of locally compact quantum groups: of Masuda, Nakagami and Woronowicz [27] and of Kustermans and Vaes [24]. The modern state of this field is reflected by collective monograph [17].

The theories of Kac algebras and of locally compact quantum groups have both two common shortcomings. First, to define the dual algebra, one needs to know the analogue of Haar measure, and its existence is an axiom, whereas for a group it follows from other axioms. Second, the objects considered are not Hopf algebras as they are defined in algebra, though close to this notion. In our opinion, at least the first problem can be resolved, if one exits the framework of Banach algebras. In this paper we propose a duality construction which is valid for a small class of groups (SIN-groups, see below) but does not require the Haar measure or weight to define the dual algebra.

This approach is inspired by a recent work by Akbarov [2] who constructs a duality for complex Lie groups with algebraic component of identity. It should be noted that in this theory the group algebras are not Banach, but are more convenient than Banach algebras from the algebraic point of view. In particular, they are Hopf algebras in the algebraic sense. The transition to the dual algebra is based on the notion of Arens–Michael envelope (the smallest algebra with submultiplicative seminorms containing the given one), instead of distinguishing a special regular representation with the help of a Haar functional.

In the present paper the main objects are pro- C^* -algebras (called also locally C^* -algebras, see [19], [14], [30]), that is, inverse, or projective, limits of C^* -algebras. Duality construction is based on the notion of a C^* -envelope, which can be axiomatically defined as the smallest pro- C^* -algebra containing the given one or explicitly as the completion of the given algebra with respect to all continuous C^* -seminorms.

For a Hopf pro- C^* -algebra A (see definition in section 6) the dual algebra is defined as follows. The conjugate space A^* , taken with topology of uniform convergence on compact subsets of A , has a natural algebra structure, so we can define its C^* -envelope $(A^*)^\diamond$. This is the dual algebra \widehat{A} of A . The second dual algebra is then $\widehat{\widehat{A}} = ((\widehat{A})^*)^\diamond$, and A is reflexive if $\widehat{\widehat{A}} = A$. In the first part of the paper (sections 2-4) we prove that the algebra $C(G)$ of continuous functions on a SIN-group G is reflexive in this sense.

Recall that G is called a SIN-group if it has a base of invariant neighborhoods of identity. This class includes all Abelian, compact and discrete groups. The diagram below illustrates the duality construction in the case of a group. We start with the algebra $C(G)$ of all continuous functions on G . Then we pass to its conjugate space, i.e. the space of linear continuous functionals on $C(G)$, which is equal to the algebra of all compactly supported measures $M_0(G)$. The C^* -envelope $\widehat{C}(G)$ of $M_0(G)$ is called the dual algebra of $C(G)$. After preparatory work in sections 2 and 3, in theorem 4.5 we prove that the second dual algebra in this sense is again $C(G)$:

$$\begin{array}{ccc}
 C(G) & \xrightarrow{\quad * \quad} & M_0(G) \\
 \uparrow C^*-\text{env} & & \downarrow C^*-\text{env} \\
 \mathcal{K}(G) & \xleftarrow{\quad * \quad} & \widehat{C}(G)
 \end{array} \tag{1}$$

Section 5 gives a structural description of $\widehat{C}(G)$, based on the structural theorem for SIN-groups.

The main result (theorem 6.10) is that there exists a category \mathfrak{SM} of reflexive Hopf pro- C^* -algebras and that it contains the algebras $C(G)$ on all SIN-groups G and their dual algebras. In section 8 we show that the C^* -algebra of the quantum group $SU_q(2)$ also belongs to \mathfrak{SM} .

At the same time, a Hopf pro- C^* -algebra is not a Hopf algebra in strictly algebraic sense, since its multiplication is defined on the projective tensor product while the comultiplication acts to the maximal C^* -tensor product. In section 7 we define a narrower category of strict Hopf pro- C^* -algebras, which complies with the algebraic notion of a Hopf algebra. This latter category also possesses a duality functor and includes all Moore groups (groups with finite-dimensional irreducible representations).

2. C^* -ENVELOPES

Our main objects are pro- C^* -algebras, or locally C^* -algebras, as they are also called. The theory of these algebras was developed by J. Inoue [19], M. Fragoulopoulou [14], C. Phillips [30] and other authors.

Definition 2.1. A C^* -seminorm on an algebra A with involution is such a seminorm p that $p(x^*x) = p(x)^2$ for all $x \in A$. We say also that p possesses the C^* -property.

This definition says nothing about continuity of multiplication and involution, but due to a theorem of Sebastiy n [32] every C^* -seminorm p automatically satisfies $p(xy) \leq p(x)p(y)$ and $p(x^*) = p(x)$ for any $x, y \in A$. This means that multiplication and involution are automatically continuous (with respect to p).

Definition 2.2. A *pro- C^* -algebra* is a complete Hausdorff topological algebra A (with jointly continuous multiplication) with involution such that its topology is generated by a family of C^* -seminorms.

Every pro- C^* -algebra A is equal to the inverse, or projective, limit of (Banach) C^* -algebras A_p over all p , where A_p is obtained as the p -completion of the quotient algebra $A/\ker p$ [30, Proposition 1.2].

Next we will give a definition of C^* -envelopes of topological algebras, which are in a sense smallest pro- C^* -algebras containing a given one. Other authors [19, 14] define C^* -envelopes only for topological $*$ -algebras with submultiplicative seminorms (Arens-Michael algebras). This is done to have a correspondence between representations of such an algebra A and its C^* -envelope. But we are not going to study the representations of A , and therefore it is not necessary to impose any additional conditions on it. We require that A is unital, but the multiplication and involution, in principle, can be even discontinuous in the initial topology. The following notations will be kept throughout the paper.

Notations 2.3. Let A be a unital algebra with involution, endowed with some locally convex topology. Denote by $\mathcal{P}(A)$ the set of all continuous C^* -seminorms on A . For every $p \in \mathcal{P}(A)$ the kernel $\ker p$ is a $*$ -ideal in A , so $A/\ker p$ is a normed algebra with the quotient norm \bar{p} of p . Its completion with respect to \bar{p} will be denoted by $C_p^*(A)$, or simply A_p when there can be no confusion. This is a (Banach) C^* -algebra. The canonical mapping $A \rightarrow C_p^*(A)$ will be denoted by i_p .

Lemma 2.4. *The algebras A_p with pointwise ordering on $\mathcal{P}(A)$ form an inverse spectrum of C^* -algebras.*

Proof. First of all, $\mathcal{P}(A)$ is directed by the pointwise ordering \leq : for $p, q \in \mathcal{P}(A)$ one may put $r = \max(p, q)$, then $p \leq r$, $q \leq r$ and C^* -property holds for r .

If $p, q \in \mathcal{P}(A)$ and $p \leq q$, then the map $\pi_p^q : i_q(A) \rightarrow i_p(A)$, $i_q(x) \mapsto i_p(x)$, is continuous on $i_q(A)$ and can be thus extended to A_q . Denote this extension still π_p^q .

Let now $p \leq q \leq r$. Obviously on $i_r(A)$ the maps π_p^r and $\pi_p^q \circ \pi_q^r$ coincide and map $i_r(x)$ to $i_p(x)$. As $i_r(A)$ is dense in A_r , these maps must also coincide on the entire algebra A_r . \square

Definition 2.5. The *C^* -envelope* A^\diamond of an algebra A is the inverse limit of the algebras A_p over $p \in \mathcal{P}(A)$.

The C^* -envelope can be alternatively defined as the completion of A/E with respect to all continuous C^* -seminorms, where E is the common kernel of $p \in \mathcal{P}(A)$. Any algebra A is continuously, but not always injectively mapped into its envelope A^\diamond . It may happen that there are no other C^* -seminorms except zero; then $A^\diamond = \{0\}$. Note that for the unit 1 of A , $p(1)^2 = p(1^*1) = p(1)$, so that $p(1)$ must be either 0 or 1. But if it is 0, then p is identically zero, so $p(1) = 1$ for every nontrivial seminorm.

Remark 2.6. The C^* -envelope has the following universal property. If $\varphi : A \rightarrow B$ is a continuous $*$ -homomorphism to a C^* -algebra B , then it may be uniquely extended to a $*$ -homomorphism $\bar{\varphi} : A^\diamond \rightarrow B$. From this it follows that if $\varphi : A \rightarrow B$ is a continuous $*$ -homomorphism of topological algebras, it may be extended to a $*$ -homomorphism $\bar{\varphi} : A^\diamond \rightarrow B^\diamond$.

When working with groups, we will use two main pro- C^* -algebras. First of them is $C(G)$, the algebra of all continuous functions on G in the topology of uniform convergence on compact sets. The second one is the C^* -envelope of the algebra $M_0(G)$ of all compactly supported measures. Note that $M_0(G)$ is the conjugate space to $C(G)$ (i.e. the space of linear continuous functionals on $C(G)$). This and other conjugate spaces we consider in special topology—of uniform convergence on totally bounded sets. If one always considers conjugate spaces in this topology, almost all classical spaces become reflexive, in particular, every Banach or Fréchet space, or the spaces $C(G)$ and $M_0(G)$ for a locally compact group G . The theory of spaces reflexive in this sense was started by Smith [35] and Brauner [6], and an extensive theory including applications to topological algebra has been developed by Akbarov [1, 2]. Following the terminology of the last author, we call these spaces *stereotype*. For convenience we pinpoint this definitions:

Notations 2.7. If X is a locally convex space, X^* denotes the space of all linear continuous functionals on X in the topology of uniform convergence on totally bounded sets. If X is isomorphic to $(X^*)^*$, X is called a *stereotype space*.

Thus, $M_0(G) = C(G)^*$. It may be noted that for all stereotype spaces the following two topologies on the conjugate space coincide: of uniform convergence on totally bounded and on compact sets. This is true, in particular, for $C(G)$ and its conjugate space $M_0(G)$. We will use the following properties of the space $M_0(G)$:

Proposition 2.8. *Let G be a locally compact group. Then:*

- (i) *G is homeomorphically imbedded into $M_0(G)$ via delta-functions, $t \mapsto \delta_t$;*
- (ii) *the linear span of delta-functions is dense in $M_0(G)$.*

Proof. It is known that the mapping of G as delta-functions is continuous [1, Theorem 10.11]. Since continuous functions on G separate points of G , this map is injective. Finally, it is well-known that a continuous injection of a locally compact space is a homeomorphism.

The conjugate space to $M_0(G)$ is $C(G)$ [1, Example 10.7] with natural pairing, what means that the values of a linear continuous functional f on $M_0(G)$ are completely determined by its values on delta-functions; this proves (ii). \square

Notations 2.9. The C^* -envelope $M_0(G)^\diamond$ of $M_0(G)$ will be denoted by $\widehat{C}(G)$. The set $\mathcal{P}(M_0(G))$ will be also denoted by $\mathcal{P}(G)$. For $p \in \mathcal{P}(G)$ we denote $C_p^*(G) = C_p^*(M_0(G))$, and also $C_\pi^*(G)$ is p is the norm of a representation π . Thus, $\widehat{C}(G) = \varprojlim_p C_p^*(G)$. Note that we do not use the reduced C^* -algebra of G , and the notation $\widehat{C}_r(G)$, if used, means just the C^* -algebra associated to a seminorm $r \in \mathcal{P}(G)$. If G is an Abelian or compact locally compact group, \widehat{G} denotes its dual group or dual space respectively.

In our scheme, $\widehat{C}(G)$ is the dual algebra to $C(G)$. Later we will show that this procedure of passing to the C^* -envelope of the conjugate space returns us to $C(G)$, if G is a SIN-group.

Every C^* -seminorm $p \in \mathcal{P}(G)$ generates a $*$ -homomorphism of $M_0(G)$ into a C^* -algebra, which may be further mapped into the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H . Let us denote this homomorphism $\psi : M_0(G) \rightarrow \mathcal{B}(H)$. The unit δ_e of $M_0(G)$ is mapped into a projection, but one can assume that it is mapped into 1_H , reducing H if necessary. This reduction will not change p . Now, as $\delta_t * \delta_t^* = \delta_e$, then $\psi(\delta_t)\psi(\delta_t)^* = 1_H$, i.e. all operators $U_t = \psi(\delta_t)$ are unitary. We see that p generates a unitary representation of G ; it is important to note that it is norm continuous.

Conversely, every norm continuous representation of G generates (via delta-functions) a non-degenerate norm continuous representation of $M_0(G)$, and consequently, a C^* -seminorm on $M_0(G)$ [1, 10.12].

Two representations of G may not be equivalent even if they generate the same seminorm, for example, π and $\pi \oplus \pi$.

Now we will describe $\widehat{C}(G)$ in the particular cases of Abelian and compact groups.

Theorem 2.10. *Let G be an Abelian locally compact group, and let \widehat{G} be its dual group. Then $\widehat{C}(G)$ coincides with the algebra $C(\widehat{G})$, taken with the topology of uniform convergence on compact sets.*

Proof. Denote for brevity $A = M_0(G)$. It is evident that for every compact set $K \subset \widehat{G}$ the seminorm

$$p_K(\mu) = \max_{t \in K} |\hat{\mu}(t)|$$

has the C^* -property (here $\hat{\mu}$ is the Fourier transform of μ). We show that every other C^* -seminorm p equals p_K for some compact $K \subset \widehat{G}$.

For every p , A_p is a unital C^* -algebra. As it is commutative, it is isomorphic to the function algebra $C(\Omega)$ on a compact space Ω . Every point $\omega \in \Omega$ defines a continuous character of the algebra A_p ; and since the map $A \rightarrow A_p$ is continuous, the restriction of ω onto A is also a continuous character of A .

As it was said before, all such characters may be identified with the points of the dual group \widehat{G} [1, 10.12]. Thus, we get a homeomorphic imbedding $\psi : \Omega \rightarrow \widehat{G}$, with

$$p(\mu) = \max_{\omega \in \Omega} |\mu(\omega)| = \max_{t \in \psi(\Omega)} |\hat{\mu}(t)|,$$

i.e. $p = p_{\psi(\Omega)}$. This proves the theorem. \square

To describe $\widehat{C}(G)$ in the compact case, we need the following [34, Corollary 2]:

Lemma 2.11. *In the decomposition of a norm continuous representation T of a compact group G into irreducible representations there may be only a finite number of non-equivalent representations.*

Theorem 2.12. *For a compact group G algebra $\widehat{C}(G)$ is equal to the direct product of the matrix algebras $C_\pi^*(G)$ over all irreducible representations π .*

Proof. By lemma 2.11, for every π we have a finite set $s(\pi) \subset \widehat{G}$ such that $\pi = \bigoplus_{j \in s(\pi)} \pi_j$, and hence $C_\pi^*(G) = \prod_{j \in s(\pi)} C_{\pi_j}^*(G)$. Conversely, every finite set generates a norm continuous representation. Next we show that if $p = \|\pi\|$, $q = \|\tau\|$ and $p \leq q$, then $s(\pi) \subset s(\tau)$. This is sufficient to prove for an irreducible representation $\pi \in \widehat{G}$. Since $p \leq q$, we have an epimorphism $R : C_\tau^*(G) \rightarrow C_\pi^*(G)$. Here $C_\pi^*(G)$ and all $C_{\tau_j}^*(G)$, $j \in s(\tau)$, are just full matrix algebras, and simple reasoning shows that the restriction

of R onto $C_{\tau_j}^*(G) \times \prod_{k \neq j} \{0\}$ is an isomorphism for one j and zero for others. This means that π is equivalent to τ_j .

Thus, $\widehat{C}(G)$ is equal to the inverse limit of finite direct products of $C_\pi^*(G)$ over $\pi \in \widehat{G}$, ordered by inclusion, but this is the direct product of all $C_\pi^*(G)$ over $\pi \in \widehat{G}$. \square

3. THE SPACE OF COEFFICIENTS AND SIN-GROUPS

Now we will discuss the conjugate space $\mathcal{K}(G) = (\widehat{C}(G))^*$ to the space $\widehat{C}(G)$. As G is continuously imbedded into $M_0(G)$, $t \mapsto \delta_t$, every functional $f \in \mathcal{K}(G)$ generates a continuous function on the group G . Moreover, if two functionals are equal on δ -functions, then by density they are equal also on $M_0(G)$ and on the entire algebra $\widehat{C}(G)$. We identify therefore f with the function on G defined in this way and represent $\mathcal{K}(G)$ as a subalgebra in $C(G)$. As usual, we endow $\mathcal{K}(G)$ with the topology of uniform convergence on totally bounded sets in $\widehat{C}(G)$.

Recall that an inverse limit $X = \varprojlim X_\alpha$ is called *reduced* if the canonical projection of X is dense in every X_α . By definition the inverse limit $A = \varprojlim A_p$, $p \in \mathcal{P}(A)$, is reduced. We will need the following simple result, close to a classical one [33, Theorem 4.4]:

Lemma 3.1. *Let $X = \varprojlim X_\alpha$ be a reduced inverse limit of locally convex spaces. Then, as a set, $X^* = \cup X_\alpha^*$, and all inclusions $X_\alpha^* \hookrightarrow X^*$ are continuous. If a set $B \subset X^*$ is dense in every X_α^* then it is dense in X^* .*

Proof. The equality $X^* = \cup X_\alpha^*$ is classical [33, Theorem 4.4]. The inclusion $X_\alpha^* \hookrightarrow X^*$ is given by the composition with the canonical projection $\pi_\alpha : X \rightarrow X_\alpha$. To prove that it is continuous, take a totally bounded set $K \subset X$. Then $\pi_\alpha(K)$ is also totally bounded, so that $p_K \leq p_{\pi_\alpha(K)}$. It follows that the topology of X^* is not stronger than the inductive limit topology.

It is easy to see that the set B is dense in X in the inductive limit topology, and the density statement follows. \square

Let us denote $\mathcal{K}_p(G) = (C_p^*(G))^*$, then it follows that $\mathcal{K}(G) = \cup \mathcal{K}_p(G)$.

Proposition 3.2. *$\mathcal{K}(G)$ has the following properties:*

- (1) $\mathcal{K}(G)$ is equal to the space of the coefficients of all norm-continuous unitary representations of G ;
- (2) $\mathcal{K}(G)$ contains the constant 1;
- (3) all functions $f \in \mathcal{K}(G)$ are bounded;
- (4) $\mathcal{K}(G)$ is contained in the linear span of all positive-definite functions on G .

Proof. Only first statement needs proof. Every $f \in \mathcal{K}_p(G)$ may be represented as a linear combination of positive functionals on $C_p^*(G)$. In their turn, they are the coefficients of representations of the algebra $C_p^*(G)$, and this is the same as norm continuous representations G . \square

Example 3.3. Discrete groups. If G is discrete, algebras $C_p^*(G)$ correspond to *all* representations of G , as they are all continuous. In particular, consider the regular representation on the space $L_2(G)$: $U_s \varphi(t) = \varphi(s^{-1}t)$. Take $\varphi = I_e$, $\psi = I_s$ (indicator

functions of the points e and s), then the function

$$f(t) = \langle U_t \varphi, \psi \rangle = \langle I_t, I_s \rangle = I_s(t)$$

belongs to $\mathcal{K}(G)$. Thus, $\mathcal{K}(G)$ contains all finitely supported functions on G . In general the regular representation is not norm continuous and $\mathcal{K}(G)$ does not contain its coefficients.

Lemma 3.4. *Linear span of the coefficients of irreducible norm continuous representations is dense in $\mathcal{K}(G)$.*

Proof. It is known that the linear span of pure states on $C_p^*(G)$ is weakly dense in $\mathcal{K}_p(G)$, and the representation corresponding to a pure state is irreducible [9]. As the unit ball B_p of $\mathcal{K}_p(G)$ is compact in the topology of uniform convergence on compact sets in $C_p^*(G)$ [1, Example 4.6], this topology coincides with the weak topology on B_p . Thus, the linear span of pure states is dense in every B_p and therefore in every $\mathcal{K}_p(G)$. By lemma 3.1 it is also dense in $\mathcal{K}(G)$. \square

Lemma 3.5. *$\mathcal{K}(G)$ is closed under pointwise multiplication and complex conjugation.*

Proof. Take $f, g \in \mathcal{K}(G)$ and prove that $f \cdot g \in \mathcal{K}(G)$. There exist C^* -seminorms p, q such that $f \in \mathcal{K}_p(G)$, $g \in \mathcal{K}_q(G)$. Consider any C^* -tensor product $\mathcal{C} = C_p^*(G) \otimes C_q^*(G)$ of the algebras $C_p^*(G)$ and $C_q^*(G)$. The mapping $U : G \rightarrow \mathcal{C}$, $t \mapsto i_p(\delta_t) \otimes i_q(\delta_t)$ (see notations 2.3), is a unitary representation of G in the Hilbert space where \mathcal{C} acts non-degenerately. It generates a representation of $M_0(G)$ and thus a continuous C^* -seminorm r on $M_0(G)$, so that $\mathcal{C} = C_r^*(G)$.

Any continuous linear functional on a C^* -algebra may be represented as a linear combination of states [20, 4.3.7], thus we can assume that f and g are states. Then there exists [20, 11.3] a product state τ on \mathcal{C} such that $\tau(x \otimes y) = f(x) \cdot g(y)$ for all $x \in C_p^*(G)$, $y \in C_q^*(G)$. For $t \in G$ we have then $\tau(t) = \tau(i_r(\delta_t)) = f(t) \cdot g(t)$, so $fg \in \mathcal{K}(G)$ and we have proved the first statement.

The second statement is proved exactly as in general theory (e.g. [18, 27.26]). \square

Our aim is the commutative duality diagram (1), and for this we need that $\mathcal{K}(G)^\diamond = C(G)$. To have this equality, it is necessary that $\mathcal{K}(G)$ separates points of G . Probably it is also sufficient, but instead of studying this question we restrict ourselves to a slightly narrower class of SIN-groups, i.e. groups with a basis of invariant neighborhoods of identity (U is invariant if $gUg^{-1} = U$ for all $g \in G$). This class includes all Abelian, compact and discrete groups and should be close to the maximal class.

SIN-groups were introduced and studied in detail by Grosser and Moskowitz [16]. A SIN-group can be equivalently defined as a group on which left and right uniform structures are equivalent, in particular, these groups are always unimodular. For connected groups, this class coincides with that of MAP-groups (maximally almost periodic, or groups on which finite-dimensional unitary representations separate points) and Z-groups (such that the quotient group over the center is compact). A connected SIN-group is a direct product of \mathbb{R}^n by a compact group. There is a complete structure theorem for SIN-groups [16]: a group G is [SIN] if and only if it is an extension

$$1 \rightarrow N = V \times K \rightarrow G \rightarrow D \rightarrow 1, \quad (2)$$

where D is discrete, $V \times K$ is a direct product, $V \simeq \mathbb{R}^n$, K is compact, and both K, V are normal in G . We will use these notations further.

Having this decomposition, we naturally use the Mackey's construction of induced representations [26]. Mackey machine requires the group to be separable, but when inducing from an open subgroup, this is not necessary. Let π be a unitary representation of N in a Hilbert space H . There are two realization of the induced representation T . The first one acts on the space $L_2(G, H)$ of square-summable H -valued functions such that $f(\xi g) = \pi(\xi)f(g)$ for all $\xi \in N, g \in G$. The action is given by $T_g f(h) = f(hg)$.

In the second realization, T acts on $L_2(D, H)$. We fix any map $s : D \rightarrow G$ such that $Ns(x) = x$, then T acts as

$$(T_g f)(x) = \pi(s_x g s_{xg}^{-1})f(xg). \quad (3)$$

Lemma 3.6. *If a unitary representation π of N is norm continuous, then the induced representation T of G is also norm continuous.*

Proof. For proof, we use the realization (3) of T on $L_2(D, H)$. Take $\varepsilon > 0$. By assumption there is a neighborhood of identity U in N , such that $\|\pi(g) - 1_H\| < \varepsilon$ when $g \in U$. Since N is open, U is at the same time a neighborhood of identity in G . Since G is a SIN-group, there exists another neighborhood of identity W such that $gWg^{-1} \subset W \subset U$ for all $g \in G$. Take now $f \in L_2(D, H)$:

$$\|T_g f - f\|^2 = \sum_{x \in D} \|\pi(s_x g s_{xg}^{-1})(f(xg)) - f(x)\|^2.$$

If $g \in W \subset N$, then $xg = x$, so that

$$\|T_g f - f\|^2 = \sum_{x \in D} \|\pi(s_x g s_x^{-1})(f(x)) - f(x)\|^2 \leq \sum_{x \in D} \|\pi(s_x g s_x^{-1}) - 1_H\|^2 \|f(x)\|^2.$$

Moreover, $s_x g s_x^{-1} \in W$, so that $\|\pi(s_x g s_x^{-1}) - 1_H\| < \varepsilon$, and it follows that $\|T_g f - f\| < \varepsilon \|f\|$, what proves the lemma. \square

The following lemma states that N is a Moore group [29, Theorems 12.4.16 and 12.4.28].

Lemma 3.7. *All irreducible representations of the group $N = \mathbb{R}^n \times K$, where K is a compact group, are finite-dimensional.*

Lemma 3.8. *Linear combinations of the functions of the type $f(v)g(k)$, where $f \in \mathcal{K}(V)$, $g \in \mathcal{K}(K)$, span a dense subspace in $\mathcal{K}(N)$.*

Proof. If ρ, σ are representations of V and K in spaces H_1, H_2 respectively, one can define a representation π of N in $H_1 \oplus H_2$ as $\pi(v, \varkappa)(x_1, x_2) = (\rho(v)x_1, \sigma(\varkappa)x_2)$. Then all functions $f \in \mathcal{K}(V)$ and $g \in \mathcal{K}(K)$ will be coefficients of π and thus belong to $\mathcal{K}(N)$. By lemma 3.5, $f(v)g(\varkappa) \in \mathcal{K}(G)$ as well.

Now we prove that the linear span of such products is dense in $\mathcal{K}(N)$. Let $F(g) = \langle \pi(g)\xi, \eta \rangle$ be a coefficient of a representation π of N . Restrictions $\rho = \pi|_V$, $\sigma = \pi|_K$ are representations of the groups V and K respectively. Then F may be represented in the following form:

$$F(g) = F(v\varkappa) = \langle \pi(v\varkappa)\xi, \eta \rangle = \langle \rho(v)\sigma(\varkappa)\xi, \eta \rangle = \langle \sigma(\varkappa)\xi, \rho(v)^*\eta \rangle.$$

By lemma 3.4 we may assume that π is irreducible, and then by lemma 3.7 it is finite-dimensional. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of H , then

$$F(g) = \sum_{i=1}^n \langle \sigma(\kappa)\xi, e_i \rangle \langle e_i, \rho(v)^*\eta \rangle = \sum_{i=1}^n f_i(v)g_i(\kappa),$$

where $f_i(v) = \langle \rho(v)e_i, \eta \rangle$ and $g_i(\kappa) = \langle \sigma(\kappa)\xi, e_i \rangle$ are coefficients of ρ and σ respectively. This proves the lemma. \square

Let I_X denote the indicator function of a set X .

Lemma 3.9. *The space $\mathcal{K}(G)$ contains all functions of the type $\varphi(g) = I_{tN}f(gt^{-1})$, where $t \in G$, $f \in \mathcal{K}(N)$. Conversely, if $I_{tN}f(gt^{-1}) \in \mathcal{K}(G)$, then $f \in \mathcal{K}(N)$.*

Proof. Let $f(x) = \langle \pi(x)\xi, \eta \rangle$ be a coefficient of a representation π of N , and let T be the induced representation of G acting by formula (3). For $t \in G$ let $\nu \in N$ be such that $\nu t = s_t$ (recall that N is a normal subgroup and its left and right cosets coincide). Put $\zeta = \pi(\nu)^*\xi$ and define functions $u = \delta_t\zeta$, $v = \delta_e\eta$ in $L_2(D, H)$. Then

$$\langle T_g u, v \rangle = \sum_{r \in D} \langle T_g u(r), v(r) \rangle = \langle (T_g u)(e), \eta \rangle = \langle \pi(g s_g^{-1})u(gN), \eta \rangle.$$

This differs from zero only if $gN = tN$. In this case $s_g = s_t = \nu t$, and $\langle T_g u, v \rangle = \langle \pi(g s_t^{-1})\zeta, \eta \rangle = \langle \pi(gt^{-1})\pi(\nu)^{-1}\zeta, \eta \rangle = \langle \pi(gt^{-1})\xi, \eta \rangle = f(gt^{-1})$, whence $\langle T_g u, v \rangle = \varphi(g)$. This proves the first statement.

Let now $\varphi(g) = I_{tN}f(gt^{-1}) = \langle \pi(g)\xi, \eta \rangle$ be a coefficient of the representation π . For $g = \nu t \in tN = Nt$ we have $\varphi(g) = \langle \pi(\nu t)\xi, \eta \rangle = \langle \pi(\nu)\xi, \pi(t)^*\eta \rangle = f(\nu)$. We see that $f(\nu)$ is a coefficient of the restriction of π onto N (which is a representation of N in the same space). \square

Corollary 3.10. *On any SIN-group G ,*

- (1) *norm continuous unitary representations separate points;*
- (2) *the algebra $\mathcal{K}(G)$ separates points.*

Proof. Due to proposition 3.2, these statements are equivalent. By lemma 3.7, the subgroup N is a Moore group, so finite-dimensional unitary representations separate its points. All of them are norm continuous. Now the statements follow directly from lemma 3.9. \square

4. DUALITY DIAGRAM FOR SIN-GROUPS

In this section we prove the main theorem 4.5. We remind that a character of an algebra is assumed to be nonzero. G is in this section a SIN-group.

Theorem 4.1. *Let A be a commutative unital algebra with involution, which is a locally convex space. Let X be the set of involutive continuous characters of A , in the weak-* topology of the dual space. Then the C^* -envelope A^\diamond of A is $C(X)$.*

Proof. For every element $a \in A$, $t \mapsto t(a)$ is by definition a continuous function on X . If $K \subset X$ is compact, then $p(a) = \sup_{\varphi \in K} |\varphi(a)|$ is a C^* -seminorm. Conversely, let p be a continuous C^* -seminorm on A . Then A_p is a commutative C^* -algebra, thus isomorphic to $C(K)$, where K is the set of its characters in the weak-* topology. If i_p is the canonical map from A to A_p , then i_p^* maps K to X , and it is easy to see that i_p^*

is continuous and injective, so that it is a homeomorphism. Thus, A^\diamond and $C(X)$ are inverse limits of equal systems of algebras:

$$A^\diamond = \varprojlim A_p = \varprojlim C(K) = C(X).$$

□

Lemma 4.2. *Let G be a compact group. For every involutive character χ of $\mathcal{K}(G)$ there is $t \in G$ such that $\chi(f) = f(t)$ for all $f \in \mathcal{K}(G)$.*

Proof. Among the representations corresponding to seminorms p there are, in particular, all irreducible unitary continuous representations. It follows that the algebra $\mathcal{K}(G)$ contains their coefficients, i.e. the classical space $\mathfrak{T}(G)$ of trigonometric polynomials [18, 27.7].

Let χ be an involutive character of the algebra $\mathcal{K}(G)$. Its restriction onto $\mathfrak{T}(G)$ is also an involutive character, and it is known [18, 30.5] that then $\chi(f) = f(t)$ for some point $t \in G$ and all $f \in \mathfrak{T}(G)$. As $\mathfrak{T}(G)$ is dense in $\mathcal{K}(G)$ by lemma 3.4, this is true for all $f \in \mathcal{K}(G)$ as well. □

Lemma 4.3. *In the notations (2), every involutive continuous character χ of the algebra $\mathcal{K}(N)$ is of the type $\chi(f) = f(t)$ for some point $t \in N$.*

Proof. By lemma 3.8, algebras $\mathcal{K}(V)$ and $\mathcal{K}(K)$ may be considered as subalgebras in $\mathcal{K}(N)$ (since each of them contains the constant 1 function). Restrictions of φ onto $\mathcal{K}(V)$ and $\mathcal{K}(K)$ are characters of these algebras. By lemmas 2.10, 4.2 there are $v_0 \in V$, $\varkappa_0 \in K$ such that $\varphi(f) = f(v_0)$ for all $f \in \mathcal{K}(V)$, $\varphi(f) = f(\varkappa_0)$ for all $f \in \mathcal{K}(K)$. Consequently, $\varphi(f) = f(v_0\varkappa_0)$ for all f in $\mathcal{K}(V) \cdot \mathcal{K}(K)$, and by lemma 3.8, since φ is continuous, for all $f \in \mathcal{K}(G)$ as well. □

Theorem 4.4. *The set of involutive continuous characters of the algebra $\mathcal{K}(G)$ may be identified with the points of the group G .*

Proof. Let $\varphi : \mathcal{K}(G) \rightarrow \mathbb{C}$ be a character. As $I_{tN}^2 = I_{tN}$ for any $t \in G$, then $\varphi(I_{tN})$ equals 0 or 1. If $sN \neq tN$, then $I_{sN} + I_{tN}$ is also an idempotent, so that $\varphi(I_{sN} + I_{tN})$ is also 0 or 1. From this it follows that $\varphi(I_{tN})$ differs from zero on one coset t_0N only.

Let us denote ${}^t f(x) = f(t^{-1}x)$. Let $\psi(f) = \varphi({}^{t_0}f)$, then ψ is also a character of $\mathcal{K}(G)$, and $\psi(I_N) = 1$. The restriction of ψ onto $\mathcal{K}(N)$ is a character of this algebra. By lemma 4.3 $\psi(f) = f(\nu)$ for some $\nu \in N$. Then $\varphi(f) = \psi({}^{t_0^{-1}}f) = f(t_0\nu)$ for translations of $\mathcal{K}(N)$, and by lemma 3.9 this is the same as all functions with supports in t_0N . Finally, as $\varphi(f) = \varphi(I_{t_0N}f)$ for any $f \in \mathcal{K}(G)$, the equality $\varphi(f) = f(t_0\nu)$ holds for all $f \in \mathcal{K}(G)$. □

This theorem is close to the theorem of Eymard [13], which states that on the algebra of coefficients of the regular representation of a locally compact group G , the only characters are the points of G . But our theorem does not follow from Eymard's, as the regular representation is continuous on a discrete group only. And also, theorem 4.4 is not valid for a general locally compact group.

We can now return to the diagram (1):

Theorem 4.5. *For any SIN-group G , the following diagram commutes:*

$$\begin{array}{ccc} C(G) & \xrightarrow{\quad \star \quad} & M_0(G) \\ \uparrow C^*-\text{env} & & \downarrow C^*-\text{env} \\ \mathcal{K}(G) & \xleftarrow{\quad \star \quad} & \widehat{C}(G) \end{array}$$

Proof. We need only to note that it follows from theorems 4.4 and 4.1 that the C^* -envelope of the algebra $\mathcal{K}(G)$ coincides with $C(G)$. \square

5. STRUCTURE OF THE DUAL ALGEBRA

Theorems 2.10 and 2.12 give the description of the dual algebras for Abelian and compact groups. Now we describe the algebra $\widehat{C}(G)$ for a discrete group G .

Theorem 5.1. *A group G is discrete if and only if $\widehat{C}(G)$ is a Banach algebra. In this case, $\widehat{C}(G)$ is equal to the classical group C^* -algebra $C^*(G)$.*

Proof. Let G be discrete. Then any representation of G is norm continuous, thus all the C^* -seminorms are continuous on $M_0(G)$. The supremum p_{\max} of all these seminorms is finite and equal to the classical C^* -norm of $\ell_1(G)$, restricted to $M_0(G)$ (which is contained in $\ell_1(G)$ in this case). From the other side, being a C^* -norm, p_{\max} is contained in $\mathcal{P}(G)$. Thus, p_{\max} is the maximal element in $\mathcal{P}(G)$, so the inverse limit $\widehat{C}(G)$ equals [11, Corollary 2.5.12] to the corresponding Banach algebra $C_{p_{\max}}$, that is, to the classical algebra $C^*(G) = \widehat{C}(G)$. Note that the dual space $(C^*(G))^*$ is equal to the Fourier–Stieltjes algebra $B(G)$ defined by Eymard [13], but taken with a weaker topology.

Conversely, let $\widehat{C}(G)$ be a Banach algebra. Then its C^* -norm p belongs to $\mathcal{P}(M_0(G))$ and $\widehat{C}(G) = C_p$. We know that G is homeomorphically imbedded into $M_0(G)$ via delta-functions (Proposition 2.8), therefore it is sufficient to show that $p(\delta_t - \delta_e) \geq 1$ for any $t \neq e$. It is clear that $p(\delta_t - \delta_e) \geq |\varphi(t) - \varphi(e)| = |\varphi(t) - 1|$ for any state φ of any algebra C_q , $q \in \mathcal{P}(G)$, when φ is identified with a function on G . Since $\mathcal{K}(G)$ separates points of G and is contained in the linear span of positive-definite functions (proposition 3.2), we can always find a positive-definite function ψ such that $\psi(t) \neq \psi(e) = 1$; we will have automatically $|\psi(t)| \leq 1$ [9, 13.4.3]. By lemma 3.5, $\mathcal{K}(G)$ is closed under pointwise multiplication and conjugation. If $|\psi(t)| = 1$, then for some n we have $|\psi(t)^n - 1| \geq 1$; if $|\psi(t)| < 1$, we can put $\varphi_n = |\psi|^{2n}$, then $|\varphi_n(t)| \rightarrow 0$, $n \rightarrow \infty$. In both cases it follows that $p(\delta_t - \delta_e) \geq 1$. \square

Next we turn to direct products of SIN-groups and show (Corollary 5.3) that they correspond to tensor products of group algebras. As with C^* -algebras (see, e.g., [20]), there is a range of possible tensor products on pro- C^* -algebras. The maximal C^* -tensor product that we use was generalized to pro- C^* -algebras by Phillips [30]. If $A = \varprojlim A_p$, $B = \varprojlim B_q$, then $A \otimes B = \varprojlim_{\max} A_p \otimes B_q$ over pairs (p, q) directed coordinately [30, Proposition 3.2]. This property, used without further reference, may be taken as definition of the maximal tensor product \otimes_{\max} . As expected, \otimes_{\max} is associative and commutative; $A \otimes_{\max} \mathbb{C} \simeq A$ for any A .

Theorem 5.2. *Let C be an algebra with a locally convex topology and let A, B be its commuting subalgebras such that the subalgebra $\langle AB \rangle$ generated by A and B is dense in C . If every pair of continuous commuting representations of A and B in the same space may be extended to a continuous representation of C , then $C^\diamond = A^\diamond \otimes_{\max} B^\diamond$.*

Proof. Let $A^\diamond = \varprojlim A_p$, $B^\diamond = \varprojlim B_q$, then $A^\diamond \otimes_{\max} B^\diamond = \varprojlim_{\max} A_p \otimes_{\max} B_q$. For a pair (p, q) let S, T be representations of A and B such that $\|(S \otimes T)(\gamma)\| = \|\gamma\|$ for any $\gamma \in A_p \otimes_{\max} B_q$ (this is a C^* -algebra, so such representations exist). By assumption there is a representation U of C such that $U|_A = S \circ T_p$, $U|_B = T \circ T_q$. Let r_{pq} be the corresponding C^* -seminorm on C . The C^* -algebra $C_{r_{pq}}$ is the closure of $U(C)$ and, since $\langle AB \rangle$ is dense in C , also the closure of the linear span of $S(A_p)T(B_q)$. The latter algebra is isomorphic to $A_p \otimes_{\max} B_q$, so $C_{r_{pq}} \simeq A_p \otimes_{\max} B_q$. Let $\varphi_{pq} : A_p \otimes_{\max} B_q \rightarrow C_{r_{pq}}$ denote this isomorphism.

For any other seminorm r on C we have restrictions $p = r|_A$ and $q = r|_B$, and since r is defined by a pair of commuting representations, it is not greater than the norm of $A_p \otimes_{\max} B_q$, i.e. $r \leq r_{pq}$. Thus, $\mathcal{P}_{\max} = \{r_{pq}\}$ is a cofinal set in $\mathcal{P}(C)$.

Now we map an element $u = (u_{pq}) \in A^\diamond \otimes_{\max} B^\diamond$ to $\psi(u) = (\psi_{pq}(u_{pq})) \in \prod C_{r_{pq}}$. Obviously the coordinates we get agree with the order of seminorms on C , so we fall into the projection of C^\diamond from $\prod_{r \in \mathcal{P}(C)} C_r$ to $\prod_{r \in \mathcal{P}_{\max}} C_r$. Then, since every ψ_{pq} is an isomorphism and \mathcal{P}_{\max} is cofinal in $\mathcal{P}(C)$, $A^\diamond \otimes_{\max} B^\diamond$ is homeomorphic to $\psi(A^\diamond \otimes_{\max} B^\diamond)$ and to C^\diamond [11, Propositions 2.5.10 and 2.5.11]. One can easily check that it is also a $*$ -algebra homomorphism, and this proves the theorem. \square

Corollary 5.3. $\widehat{C}(G \times H) = \widehat{C}(G) \otimes_{\max} \widehat{C}(H)$ for any SIN-groups G and H .

Proof. We can apply theorem 5.2 with $A = M_0(G)$, $B = M_0(H)$ and $C = M_0(G \times H)$. Here $\langle AB \rangle$ is the linear span of all delta-functions, since $\delta_{s,e} \cdot \delta_{e,t} = \delta_{s,t}$, and so it is dense in $M_0(G \times H)$. Next, every pair of commuting representations of G and H defines a representation of $G \times H$ (with the same type of continuity); and bijection between representations of groups and their measure algebras gives us the required extension property. \square

After several lemmas, we describe algebra $\widehat{C}(G)$ in more detail. Theorem 5.8 gives decomposition of all representations of G into finite sums corresponding to different irreducible representations $\pi \in \widehat{K}$ of the compact subgroup K . It follows (corollary 5.10) that $\widehat{C}(G)$ is a direct product of algebras $C_\pi^*(G)$ each of which corresponds to a unique representation $\pi \in \widehat{K}$.

The following lemma is a particular case of [30, Proposition 3.4]:

Lemma 5.4. *For any pro- C^* -algebra A and a locally compact space M , $C(M) \otimes_{\max} A$ is isomorphic to the algebra $C(M, A)$ of continuous functions from M to A , in the topology of uniform convergence on compact sets.*

Lemma 5.5. *Let G be a SIN-group. In notations (2), for $t \in D$ denote $N_t = \{\mu \in M_0(G) : \text{supp } \mu \subset tN\}$. Then a seminorm p on $M_0(G)$ is continuous if and only if its restriction to every N_t , $t \in D$, is continuous.*

Proof. Recall that the topology of $M_0(G)$ is defined by the system of seminorms $p_L(\mu) = \sup_{f \in L} |\mu(f)|$, where L runs over all totally bounded sets in $C(G)$. A set $L \subset C(G)$ is totally bounded if and only if it is uniformly bounded and equicontinuous on every compact set $S \subset G$ [22, Theorem 7.17]. Such a set S intersects with only finitely many open cosets tN , $t \in D$; thus the said conditions for S hold iff they do for every intersection $S \cap (tN)$. But this means exactly that L is totally bounded iff the family of its restrictions $L_t = \{f|_{tN} : f \in L\}$ is totally bounded in every space $C(tN)$, $t \in D$.

The set $\bar{L} = \{f : f|_{tN} \in L_t\}$ is also totally bounded, so $p_{\bar{L}}$ is a continuous seminorm on $M_0(G)$; since $p_L \leq p_{\bar{L}}$, p_L is also continuous on $M_0(G)$. \square

Lemma 5.6. *Let G be a SIN-group with a decomposition (2). Then*

- (1) *for any seminorm $p_0 \in \mathcal{P}(N)$, there exists a seminorm $p \in \mathcal{P}(G)$ such that $p|_{\widehat{C}(N)} = p_0$;*
- (2) *the supremum of all such seminorms $p_{\max} = \sup\{p \in \mathcal{P}(G) : p|_{\widehat{C}(N)} = p_0\}$ is finite and contained in $\mathcal{P}(G)$;*
- (3) *if $q \in \mathcal{P}(G)$ and $q|_{\widehat{C}(N)} \leq p_0$, then $q \leq p_{\max}$.*

Proof. (1) Let $p_0 \in \mathcal{P}(N)$. It corresponds to a representation of N , which induces a representation T of G (lemma 3.6); then the seminorm p corresponding to T satisfies $p|_{\widehat{C}(N)} = p_0$.

(2) Let $\mu \in M_0(G)$ be a compactly supported measure, and for $t \in D$ let μ_t be restriction of μ onto tN . Since the cosets tN are open, only finite number of μ_t is nonzero. Since δ_t is a unitary element, $p(\mu_t) = p(\delta_{t^{-1}} * \mu_t)$; now we see that

$$p(\mu) \leq \sum_{t \in D} p(\mu_t) = \sum_{t \in D} p_0(\delta_{t^{-1}} * \mu_t) \equiv \bar{p}(\mu), \quad (4)$$

where the right-hand side does not depend on p . Thus, $p_{\max} \leq \bar{p} < \infty$. By lemma 5.5, \bar{p} is continuous on $M_0(G)$, so the same is true for p_{\max} . We need only to add that the supremum of a family of C^* -seminorms is also a C^* -seminorm.

(3) Let $r = \max(q, p_{\max})$. By assumption $r|_{\widehat{C}(N)} = p_0$, so by (2) $r \leq p_{\max}$. It follows that $r = p_{\max}$, that is, $q \leq p_{\max}$. \square

Lemma 5.7. *Let $\pi \in \widehat{K}$ and let χ_π^K be its character. Define a measure χ_π on G as $\chi_\pi(f) = \int_K f \chi_\pi^K$. Then χ_π is central in $M_0(G)$.*

Proof. It is a classical fact that χ_π commutes with δ_t for $t \in K$. Let now $t \in G \setminus K$, and let $\xi = \delta_t \chi_\pi \delta_t^{-1}$. Since K is normal, $\xi \in \widehat{C}(K)$. Let us compare Fourier transforms of ξ and χ_π . If $\sigma \neq \pi$, then $\sigma(\xi) = 0$, as $\sigma(\chi_\pi) = 0$. Let T_π be the representation of G induced by π . Then $T_\pi(\chi_\pi) = \text{id}$, as it may be seen from the explicit construction of T_π . It follows that $T_\pi(\xi) = T_\pi(\delta_t \chi_\pi \delta_t^{-1}) = \text{id}$. And it is easy to notice that then also $\pi(\xi) = \text{id}$. This yields $\xi = \chi_\pi$, what proves the lemma. \square

Theorem 5.8. *Every norm continuous representation T of G is a finite direct sum of representations T_π such that $T_\pi|_K$ is a multiple of a single irreducible representation $\pi \in \widehat{K}$.*

Proof. For $\mu \in M_0(G)$, put $T_\pi(\mu) = T(\chi_\pi \mu \chi_\pi)$. This is a subrepresentation of T such that $T_\pi|_K$ is a multiple of π . From lemma 2.11 it follows that T_π is nonzero for finite number of π . Since $\chi_\pi * \chi_\sigma = 0$ for $\pi \neq \sigma$, T is the direct sum of T_π . \square

Before we can formulate corollaries on the structure of $\widehat{C}(G)$, let us describe the seminorms on $\widehat{C}(N)$. In notations (2), by lemmas 5.3, 2.10 and 5.4, we have

$$\widehat{C}(N) = \widehat{C}(\mathbb{R}^n \times K) = \widehat{C}(\mathbb{R}^n) \otimes_{\max} \widehat{C}(K) = C(\mathbb{R}^n) \otimes_{\max} \widehat{C}(K) = C(\mathbb{R}^n, \widehat{C}(K)). \quad (5)$$

A defining system of seminorms on $\widehat{C}(N)$ has then the following form: for $k \in \mathbb{N}$ and $\sigma = \{\pi_1, \dots, \pi_m\} \subset \widehat{K}$

$$p_{k,\sigma}(\mu) = \sup_{|x| \leq k} \max_{\pi \in \sigma} \|\mu(x)\|_{C_\pi^*(G)}, \quad (6)$$

where $|x|$ is any fixed norm in \mathbb{R}^n .

Corollary 5.9. *For every $\sigma \subset \widehat{K}$ and $k \in \mathbb{N}$, $p_{k,\sigma}^{\max} = \max\{p_{k,\pi}^{\max} : \pi \in \sigma\}$.*

Proof. There is a representation T such that $p_{k,\sigma}^{\max} = \|T\|$. By the preceding theorem we have finite decompositions $T = \oplus T_\pi$ and $T|_K = \oplus \pi$. $T|_V$ is the multiplication in $L_2(M)$ for some compact set $M \subset \mathbb{R}^n$; easy to show that $M \subset \{x : |x| \leq k\}$. Then $\|T_\pi|_N\| \leq p_{k,\pi}$, and by lemma 5.6 (3) $\|T_\pi\| \leq p_{k,\pi}^{\max}$, what proves the statement. \square

Corollary 5.10. *For a SIN-group G , $\widehat{C}(G) = \prod_{\pi \in \widehat{K}} C_\pi$ where $C_\pi = \varprojlim_k C_{p_{k,\pi}^{\max}}^*(G)$ if $V \neq \{0\}$ and $C_\pi = C_{p_\pi^{\max}}^*(G)$ otherwise.*

Corollary 5.11. *The category $\mathfrak{P}\mathfrak{E}\mathfrak{F}$ of pro- C^* -algebras which are direct products of Fréchet spaces contains $C(G)$ and $\widehat{C}(G)$ for all SIN-groups G and is closed under tensor products. Every space in this category is stereotype.*

Proof. Due to decomposition (2), $C(G)$ is homeomorphic to $C(N)^D$ as a locally convex space. Since $C(N)$ is a Fréchet space, $C(G) \in \mathfrak{P}\mathfrak{E}\mathfrak{F}$. Every C_π in lemma 5.10 is a countable inverse limit of Banach spaces, so it is a Fréchet space, thus $\widehat{C}(G) \in \mathfrak{P}\mathfrak{E}\mathfrak{F}$. Direct product of Fréchet spaces is stereotype by [1, 4.4 and 4.20]. \square

6. HOPF PRO- C^* -ALGEBRAS

In this section we show that SIN-groups may be imbedded into a category of Hopf pro- C^* -algebras, reflexive with respect to the duality map $A \mapsto \widehat{A} = A^{\star\Diamond}$. In section 6 we prove that it contains also the $SU_q(2)$ group. We must note that these are not Hopf algebras in strict algebraic sense (definitions are given below). Therefore, in section 7 the work is continued and we introduce another category of strictly defined Hopf algebras; we show that all Moore groups, but not all SIN-groups are imbedded into this second category.

We consider only unital algebras over \mathbb{C} . General definitions of an algebra, coalgebra and a Hopf algebra in a tensor category may be found in [7]. To indicate the tensor product we use the terms \otimes -algebra, \otimes -coalgebra and so on.

For us, categories will consist of topological algebras (pro- C^* -algebras and their conjugate spaces) and respective topological tensor products. If A is a \otimes -algebra, in analytical language this means that the multiplication $\mathfrak{m} : A \times A \rightarrow A$ may be extended to a continuous map $\mathfrak{m} : A \otimes A \rightarrow A$. The notations \mathfrak{m} , ι , Δ and ε will be used for multiplication, unit, comultiplication and counit in any algebra or coalgebra within this section.

A \otimes -algebra with continuous involution $*$ will be called an *involutive \otimes -algebra*. An algebra morphism φ is a morphism of involutive algebras if $\varphi(a)^* = \varphi(a^*)$. Recall that

in a Hopf algebra with involution one requires that Δ and ε be $*$ -homomorphisms [7]. To indicate the corresponding tensor product, we use the term \otimes -Hopf $*$ -algebra.

Our main category \mathfrak{C} is that of pro- C^* -algebras with continuous $*$ -homomorphisms. In general, $*$ -homomorphisms are not automatically continuous, in contrast to Banach C^* -algebras. \mathfrak{C} is a tensor category with \otimes_{\max} .

For any locally compact group G , $C(G)$ is a \otimes_{\max} -Hopf $*$ -algebra. This is evident if we note that $C(G) \otimes_{\max} C(G) = C(G \times G)$. This is not always the case for the dual algebra $\widehat{C}(G)$. The class of groups such that $\widehat{C}(G)$ is a \otimes_{\max} -Hopf $*$ -algebra is close to Moore groups. We call a \otimes_{\max} -algebra also a *strict Hopf pro- C^* -algebra*. Motivation for this term and further properties of strict algebras are postponed till section 7.

Nevertheless, $\widehat{C}(G)$ is always a Hopf algebra in the following sense (note that here, in fact, two different tensor products are used — one for multiplication and another for comultiplication):

Definition 6.1. A *Hopf pro- C^* -algebra* is a pro- C^* -algebra which is a \otimes_{\max} -coalgebra such that its comultiplication and counit are $*$ -homomorphisms.

Every strict Hopf pro- C^* -algebra is a Hopf pro- C^* -algebra, but in general converse is not true. In the rest of the section, we define a subcategory \mathfrak{SM} of reflexive Hopf pro- C^* -algebras and a duality $A \mapsto \widehat{A} = A^{\diamond}$ on it.

The category of Hopf pro- C^* -algebras is closed under tensor products:

Lemma 6.2. *If A and B are Hopf pro- C^* -algebras, then so is $A \otimes_{\max} B$.*

Proof. All structural mappings are $*$ -homomorphisms, so they may be extended continuously to respective tensor products. \square

Let \mathfrak{C}^* denote the category of conjugate spaces to pro- C^* -algebras, with continuous linear maps as morphisms. In order to have a well-defined algebra structure on the conjugate space A^* of a pro- C^* -algebra A , we need that A be stereotype. Then we can define a tensor product $M \otimes N = (M^* \otimes_{\max} N^*)^*$ for conjugate spaces of stereotype algebras. Since it is unknown in general whether the maximal tensor product preserves the property of being stereotype, these conjugate spaces do not form a tensor category. Nevertheless, it is possible to define \otimes -algebras, coalgebras, Hopf algebras with the help of conjugate maps to structural maps in \mathfrak{C} . Further we will use the terms \otimes -algebra, \otimes -coalgebra, \otimes -Hopf $*$ -algebra in this limited sense.

Definition 6.3. Let A be a stereotype Hopf pro- C^* -algebra. Then A^* is a topological algebra with involution, and we can define its C^* -envelope $(A^*)^{\diamond}$. This algebra will be denoted \widehat{A} and will be called the *dual algebra to A* .

In general, there may be no natural structure of a Hopf algebra on \widehat{A} . We will continue to describe cases where this structure exists and show that it exists, in particular, on group algebras $\widehat{C}(G)$.

Lemma 6.4. *For every stereotype Hopf pro- C^* -algebra A , there is an injective continuous linear map $I : (\widehat{A})^* \rightarrow A$.*

Proof. First of all, composition with $i : A^* \rightarrow \widehat{A}$ of every $\varphi \in (\widehat{A})^*$ is a continuous linear functional $I(\varphi) = \varphi \circ i$ on A^* , and by duality $I(\varphi) \in A$. Next, since $i(A^*)$ is dense in \widehat{A} , this mapping $\varphi \mapsto I(\varphi)$ is injective. Further, every seminorm on A has form $p(a) = \sup_{\mu \in M} |\mu(a)|$ for a totally bounded set $M \subset A^*$. Its image $i(M) \subset \widehat{A}$ is also totally bounded, thus $\tilde{p}(\alpha) = \sup_{\mu \in i(M)} |\alpha(\mu)|$ is a continuous seminorm on \widehat{A}^* . This shows that $I : (\widehat{A})^* \rightarrow A$ is continuous. \square

This inclusion I allows to define a multiplication on $(\widehat{A})^*$. In general, $I(\varphi) \cdot I(\psi)$ may not belong to $I((\widehat{A})^*)$, but if this is the case and if multiplication, defined in this way, may be extended continuously to $(\widehat{A})^* \otimes_{\max} (\widehat{A})^*$ (this has to be checked separately in each case), then $(\widehat{A})^*$ is a \otimes_{\max} -algebra. If \widehat{A} is stereotype, then \widehat{A} acquires a natural structure of a \otimes_{\max} -coalgebra. We will say then that \widehat{A} is a Hopf pro- C^* -algebra with dual to A structure. Thus we come to the following category:

Theorem 6.5. *On the category of Hopf pro- C^* -algebras A such that A and \widehat{A} are stereotype and \widehat{A} has dual structure of a Hopf pro- C^* -algebra, $A \mapsto \widehat{A}$ is a functor.*

Proof. If $\varphi : A \rightarrow B$ is a morphism in \mathfrak{C} , then $\widehat{\varphi}$ is defined in the following natural way. The conjugate map $\varphi^* : B^* \rightarrow A^*$ is a $*$ -homomorphism of \otimes -algebras, and by universality it may be extended to a $*$ -homomorphism of respective C^* -envelopes $\widehat{\varphi} : \widehat{B} \rightarrow \widehat{A}$.

For any $\xi \in (\widehat{A})^*$ and $\beta \in B^*$ we have pairings in A, A^* and B, B^* :

$$\langle (\widehat{\varphi})^* \xi, \beta \rangle = \langle \xi, \widehat{\varphi} \beta \rangle = \langle \xi, i \varphi^*(\beta) \rangle = \langle I \xi, \varphi^* \beta \rangle = \langle \varphi(I \xi), \beta \rangle.$$

Thus, on $I((\widehat{A})^*)$ we have $\varphi I = (\widehat{\varphi})^*$. Since multiplication on $(\widehat{A})^*$ agrees with that on A , $(\widehat{\varphi})^*$ is a homomorphism, so $\widehat{\varphi}$ is a coalgebra morphism. The rest details are obvious. \square

Theorem 6.6. *If A is a stereotype \otimes_{\max} -coalgebra, then its conjugate space A^* is a \otimes -algebra. If A is a stereotype strict Hopf pro- C^* -algebra, then its conjugate space A^* is a \otimes -Hopf $*$ -algebra.*

Proof. This follows automatically from the diagrams in definitions by taking conjugate operators. \square

In the following lemma we prove that \widehat{A} is automatically a Hopf pro- C^* -algebra under certain conditions. One of requirements is that the antipode S is a $*$ -antihomomorphism, i.e. $S(a^*) = S(a)^*$ (besides the usual condition $S(ab) = S(b)S(a)$). This is equivalent to the identity $S^2 = \text{id}$, what follows from the equality $S(x)^* = S^{-1}(x^*)$ [7, 4.1].

Lemma 6.7. *Let A be a strict Hopf pro- C^* -algebra such that $(A^* \otimes A^*)^\diamond = \widehat{A} \otimes_{\max} \widehat{A}$. Then $A^{*\diamond}$ with structural maps extended by continuity from A^* is a \otimes_{\max} -coalgebra; if, in addition, $S^2 = \text{id}$ for the antipode of A , then \widehat{A} is a Hopf pro- C^* -algebra.*

Proof. By theorem 6.6, A^* is a \otimes -Hopf $*$ -algebra. Its comultiplication Δ and counit ε are $*$ -homomorphisms, and they are extended to $*$ -homomorphisms of respective C^* -envelopes by the universality property. If $S^2 = \text{id}$, then the same identity $\tilde{S}^2 = \text{id}$

holds for the antipode \tilde{S} of A^* , and from the known relations between antipode and involution [21, 1.2.7] it follows that $\tilde{S} \circ *$ is a $*$ -homomorphism: $(\tilde{S} \circ *)(a^*) = \tilde{S}(a) = \tilde{S}^{-1}(a) = (* \circ \tilde{S} \circ *)(a) = (\tilde{S} \circ *)(a)^*$. Thus, it is extended by continuity to the envelope \hat{A} . It remains to note that $\tilde{S} = \tilde{S} \circ * \circ *$ and the involution is continuous on \hat{A} . \square

Corollary 6.8. *For any SIN-group G , $\hat{C}(G)$ is a Hopf pro- C^* -algebra.*

Proof. We can apply lemma 6.7 with $M = M_0(G)$, because $C(G)$ is strict, and by theorem 5.3

$$\begin{aligned} (M_0(G) \otimes M_0(G))^\diamond &= (C(G) \otimes_{\max} C(G))^{*\diamond} = M_0(G \times G)^\diamond \\ &= \hat{C}(G \times G) = \hat{C}(G) \otimes_{\max} \hat{C}(G) = M_0(G)^\diamond \otimes_{\max} M_0(G)^\diamond. \end{aligned}$$

\square

Definition 6.9. Let \mathfrak{SIN} denote the category of pro- C^* -algebras A such that:

- (i) the dual algebra \hat{A} is well defined;
- (ii) both A and \hat{A} are stereotypic;
- (iii) $\hat{\hat{A}} = A$.

Morphisms in \mathfrak{SIN} are $*$ -algebra and \otimes_{\max} -coalgebra homomorphisms.

From the previous results follows

Theorem 6.10. *The category \mathfrak{SIN} contains the algebras $C(G)$ and $\hat{C}(G)$ for all SIN-groups G . The mapping $A \mapsto \hat{A}$ is a contravariant duality functor on \mathfrak{SIN} .*

In section 8 we will show that \mathfrak{SIN} contains also the C^* -algebra of the quantum group $SU_q(2)$.

In the rest of the section we prove that under certain conditions $\hat{}$ commutes with tensor products.

Lemma 6.11. *Let A and B be pro- C^* -algebras. If A or B is nuclear, the algebraic tensor product $A^* \otimes B^*$ is dense in $(A \otimes_{\max} B)^*$.*

Proof. Since A^*, B^* are the linear spans of states on A and B respectively, and any pair of states $\varphi \in A^*, \psi \in B^*$ defines a product state on $A \otimes_{\max} B$, we have a map from $A^* \otimes B^*$ to $(A \otimes_{\max} B)^*$. It is easy to check that this map is injective.

From $A \otimes_{\max} B = \varprojlim_{\max} (A_p \otimes_{\max} B_q)$ we have $A^* \otimes B^* = \varprojlim_{\max} (A_p \otimes_{\max} B_q)^*$. By lemma 3.1, it is sufficient to prove that $A^* \otimes B^*$ is dense in every $(A_p \otimes_{\max} B_q)^*$, so we may assume that A and B are just C^* -algebras.

Since A or B is nuclear, $A \otimes_{\max} B$ is isomorphic to their minimal (spatial) tensor product $A \otimes_{\min} B$, so we may assume that there are Hilbert spaces H_1, H_2 such that $A \subset \mathcal{B}(H_1), B \subset \mathcal{B}(H_2)$, and $a \otimes b \in A \otimes_{\min} B$ acts on $x_1 \otimes x_2 \in H = H_1 \otimes H_2$ as $ax_1 \otimes bx_2$. Then for any $u \in A \otimes_{\min} B$ we have

$$\|u\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} (ux, y) \quad (7)$$

with $x, y \in H$. Let e_α, f_β be bases in H_1 and H_2 respectively, and denote $\omega_{xy}(u) = (ux, y)$. For basic vectors, $\omega_{e_\alpha \otimes f_\beta, e_\xi \otimes f_\eta}$ is a product functional (indeed, if $u = a \otimes b$ then $(ue_\alpha \otimes f_\beta, e_\xi \otimes f_\eta) = (ae_\alpha \otimes bf_\beta, e_\xi \otimes f_\eta) = (ae_\alpha, e_\xi)(bf_\beta, f_\eta)$). If $x = \sum x_{\alpha\beta} e_\alpha \otimes f_\beta$, $y = \sum y_{\xi\eta} e_\xi \otimes f_\eta$, then clearly in (7) it is enough to take x, y such that these sums are finite. Then we have $\omega_{xy} \in A^* \otimes B^*$.

Since $A \otimes B$ is a Banach space, it is stereotype and therefore any linear continuous functional φ on $(A \otimes B)^*$ is of the type $\varphi(\omega) = \omega(u)$ for some $u \in A \otimes B$. From preceding, if $\varphi(\omega) = \omega(u) = 0$ for all $\omega \in A^* \otimes B^*$, then $u = 0$. This proves the lemma. \square

Lemma 6.12. *Let A and B be stereotype Hopf pro- C^* -algebras, and let $M = A^*$, $N = B^*$ be their conjugate \otimes -algebras. If A or B is nuclear and M or N is strict, then $(M \otimes N)^\diamond = M^\diamond \otimes N^\diamond$.*

Proof. We will prove that assumptions of theorem 5.2 hold for M, N and $M \otimes N$. First note that M and N are imbedded into $M \otimes N$ as subalgebras $M \otimes \varepsilon', \varepsilon \otimes N$, where $\varepsilon, \varepsilon'$ are the units of M and N respectively (the counits of A and B). The subalgebra generated by M and N coincides with the algebraic tensor product $M \otimes N$, which is dense in $M \otimes N$ by lemma 6.11.

Next we prove that for every $p \in \mathcal{P}(M)$, $q \in \mathcal{P}(N)$ the seminorm $p \otimes q$ of the maximal tensor product $M_p \otimes N_q$ is continuous on $M \otimes N$, this will show that every pair of representations of M and N may be extended to a representation of $M \otimes N$.

It is known that a strict algebra is also nuclear (see, e.g. [25, Theorem 2.5]). Thus, its maximal and minimal tensor products with any C^* -algebra are equal; by [4, Theorem 7] it is also topologically isomorphic to the injective Banach space tensor product. Then the norm of a tensor $u \in M_p \otimes N_q$ is equal to $\sup_{\varphi \otimes \psi} (\varphi \otimes \psi)(u)$ over functionals φ, ψ in the unit balls S_1, S_2 of M_p^* and N_q^* respectively. Any functional on M_p defines a functional on M , so we may assume that $S_1 \subset M^* = A$. Since S_1 is compact in M_p^* , this inclusion preserves its topology. Similarly, we assume that $S_2 \subset B$.

Now note that $p(\mu) = \sup_{\varphi \in S_1} |\varphi(\mu)|$ and $q(\nu) = \sup_{\psi \in S_2} |\psi(\nu)|$ for $\mu \in M$ and $\nu \in N$. Since p and q are continuous, by definition of topologies on $M = A^*$ and $N = B^*$ there exist compact sets $K \subset A$ and $L \subset B$ such that $S_1 \subset K, S_2 \subset L$, so that $p \leq p_K = \sup_{\varphi \in K} |\varphi(\mu)|$ and $q \leq p_L$. It follows that $S_1 \otimes S_2 \subset K \otimes L$ and $(p \otimes q)(u) \leq p_{K \otimes L} = \sup_{\zeta \in K \otimes L} |\zeta(u)|$. It remains only to prove a simple exercise that $K \otimes L$ is compact in $A \otimes B$.

In fact, every seminorm $r \in \mathcal{P}(A \otimes B)$ has the form $r = r_1 \otimes r_2$, where $r_1 \in \mathcal{P}(A)$, $r_2 \in \mathcal{P}(B)$. We need to prove that for every $\varepsilon > 0$ there is a finite ε -net for $K \otimes L$ with respect to r . Since K and L are compact, $C_1 = \sup_{x \in K} r_1(x) < \infty$, and $C_2 = \sup_{y \in L} r_2(y) < \infty$. Let $\delta < (2 \max\{C_1, C_2\})^{-1}$. Choose finite δ -nets $\{x_j\}_{j=1}^n$ in K with respect to r_1 and $\{y_k\}_{k=1}^m$ in L with respect to r_2 . Then $\{x_j \otimes y_k\}$ is an ε -net in $K \otimes L$. \square

Corollary 6.13. *Let G and H be SIN-groups, then $(C(G) \otimes \widehat{C}(H))^\diamond = \widehat{C}(G) \otimes C(H)$.*

Proof. We can apply lemma 6.12:

$$(C(G) \underset{\max}{\otimes} \widehat{C}(H))^{\star\Diamond} = (M_0(G) \otimes \mathcal{K}(H))^{\Diamond} = M_0(G)^{\Diamond} \underset{\max}{\otimes} \mathcal{K}(H)^{\Diamond} = \widehat{C}(G) \underset{\max}{\otimes} C(H)$$

since the algebras $C(G)$, $\mathcal{K}(H)$ are commutative, hence nuclear and strict. \square

Corollary 6.14. *Let G and H be SIN-groups, then $C(G) \underset{\max}{\otimes} \widehat{C}(H) \in \mathfrak{SM}$.*

7. STRICT ALGEBRAS AND MOORE GROUPS

In section 6 we have introduced the term *strict Hopf pro- C^* -algebra* as a synonym for a $\underset{\max}{\otimes}$ -Hopf $*$ -algebra. In this section we link these algebras to strict Banach and C^* -algebras and show that $\widehat{C}(G)$ is a strict algebra for any Moore group G (corollary 7.6). In proposition 7.7 we prove that if G is a discrete group but not Moore then $\widehat{C}(G)$ is not strict. The C^* -algebra of the quantum group $SU_q(2)$, being an analogue of a compact group, is nevertheless not strict either.

Thus, there is a subcategory of \mathfrak{SM} , self-dual with respect to $\widehat{}$, which consists of algebras A such that both A and \widehat{A} are strict Hopf pro- C^* -algebras. It contains the algebras $C(G)$, $\widehat{C}(G)$ of all Moore groups G . Interest in this category is due to the fact that it strictly complies to algebraical notion of Hopf algebras.

Multiplication on a $\underset{\max}{\otimes}$ -algebra is connected to the following properties:

Definition 7.1. A Banach algebra A is called *strict* if it is a $\check{\otimes}$ -algebra with the injective tensor product $\check{\otimes}$.

Definition 7.2. A C^* -algebra A is said to be of *bounded degree n* if all its irreducible representations are finite-dimensional and their dimensions not exceed n .

In [4, Theorem 6] it is proved that a C^* -algebra is strict if and only if it is of bounded degree, and if and only if it is also a $\underset{\min}{\otimes}$ -algebra with the minimal C^* -tensor product. There are many equivalent properties of algebras of bounded degree [25, Theorem 2.5]. Another criterion is also true:

Lemma 7.3. *A C^* -algebra A is strict if and only if it is a $\underset{\max}{\otimes}$ -algebra.*

This lemma is proved exactly in the same way as it is done by Aristov [4] for minimal tensor products. For completeness, we repeat the argument below, including proof of the following proposition which is stated as obvious in [4].

Proposition 7.4. *Let A be a C^* -algebra and $a_1, \dots, a_m \in A$ such that $a_i a_j^* = a_i^* a_j = 0$ if $i \neq j$. Then*

$$\|\sum_i a_i\| \leq \max \|a_i\|.$$

Proof. Let $S_0(a) = a$, $S_n(a) = S_{n-1}(a)^* S_{n-1}(a)$ for any $a \in A$. Then, by induction, $S(a_i)$ satisfy the same conditions as a_i :

$$\begin{aligned} S_n(a_i) S_n(a_j)^* &= S_{n-1}(a_i)^* S_{n-1}(a_i) S_{n-1}(a_j)^* S_{n-1}(a_j) = 0 \text{ if } i \neq j, \\ S_n(a_i)^* S_n(a_j) &= 0 \text{ if } i \neq j. \end{aligned}$$

These properties imply that

$$\|\sum_i S_n(a_i)\|^2 = \|\sum_{i,j} S_n(a_j)^* S_n(a_i)\| = \|\sum_i S_n(a_i)^* S_n(a_i)\| = \|\sum_i S_{n+1}(a_i)\|,$$

so that by induction

$$\left\| \sum a_i \right\|^{2^n} = \left\| \sum S_n(a_i) \right\|.$$

Now, as $\|S_n(a)\| = \|S_{n-1}(a)\|^2$, we have

$$\left\| \sum a_i \right\| \leq \left(\sum \|S_n(a_i)\| \right)^{1/2^n} = \left(\sum \|a_i\|^{2^n} \right)^{1/2^n}.$$

The right-hand side tends to $\max \|a_i\|$ as $n \rightarrow \infty$, what proves the proposition. \square

Proof of lemma 7.3. If A is strict, then its multiplication \mathbf{m} extends to a continuous operator from $A \underset{\min}{\otimes} A$ to A ; since there is a continuous map from $A \underset{\max}{\otimes} A$ to $A \underset{\min}{\otimes} A$, the multiplication is continuous on $A \underset{\max}{\otimes} A$ also.

For the inverse statement, we follow reasoning of Aristov. In [3] it is proved that a C^* -algebra is strict if and only if it is of bounded degree. Thus, we need to show that every $\underset{\max}{\otimes}$ -algebra is of bounded degree. Suppose that A is a $\underset{\max}{\otimes}$ -algebra and there is an irreducible representation π of A of dimension n . Let $\{e_i\}_{i=1}^n$ be a basis in the space of representation of π , and let $\varepsilon_i(v) = \langle v, e_i \rangle$ be the coordinate functionals. Then by [4, Lemma 1] there exist elements $a_i \in A$, $i = 1, \dots, n$ (in notations of [4], $a_i = v_{i1}$) such that

$$\begin{aligned} \|a_i\| &= 1; \\ a_i a_j^* &= a_i^* a_j = 0 \text{ if } i \neq j; \\ q\pi(a_i)q &= \pi(a_i)q = \varepsilon_i e_i, \text{ where } q = \sum_{i=1}^n \varepsilon_i e_i. \end{aligned}$$

Take $x = \sum_i a_i^* \otimes a_i \in A \underset{\max}{\otimes} A$. Then

$$\begin{aligned} q\pi(\mathbf{m}(x))q &= \sum_i q\pi(a_i^* a_i)q = \sum_i (\pi(a_i)q)^* \pi(a_i)q = \sum_i (\varepsilon_i e_i)^* \varepsilon_i e_i \\ &= \sum_i \varepsilon_i e_1 \circ \varepsilon_i e_i = \sum_i \varepsilon_i e_1 = n \varepsilon_1 e_1. \end{aligned}$$

Thus, $n = \|q\pi(\mathbf{m}(x))q\| \leq \|\mathbf{m}(x)\|$. At the same time, by proposition 7.4, $\|x\| \leq \max \|a_i^* \otimes a_i\| = 1$. This means that $\|\mathbf{m}\| \geq n$, so if \mathbf{m} is a continuous operator then A must be of bounded degree. \square

Lemma 7.5. *If G is a Moore group and $p \in \mathcal{P}(G)$, then the algebra $C_p^*(G) = C_p^*(M_0(G))$ (defined in 2.3) is of bounded degree.*

Proof. I. It is known [29, Theorem 12.4.27] that a Moore group is equal to the projective limit of Lie Moore groups. By [34, Theorem 1], every norm continuous representation of G may be factored through one of the quotient groups in this limit, so we can assume that G is itself a Lie group.

II. By [29, Theorem 12.4.27], G is a finite extension of a central group G_1 . Let Z be the center of G_1 . By [15, Theorem 4.4] G_1 is equal to the direct product $\mathbb{R}^m \times G_2$, where G_2 has a compact open normal subgroup K . Let $H = ZK$, then H is normal in G_1 and contains Z ; as it also contains $\mathbb{R}^m \times K$, it is open in G_1 . Thus G_1/H is discrete and compact, hence finite. As $G/G_1 = (G/H)/(G_1/H)$, we see that $G/H = F$ is finite.

Suppose we have proved the theorem for the subgroup H . Denote $|F| = m$. Take any $p \in \mathcal{P}(G)$ and an irreducible representation τ of $C_p^*(G)$ on a space E . As G is a Moore group, τ is finite-dimensional. By [8, Theorem 1], the restriction of $\tau \circ \pi$ onto H decomposes into at most m irreducible representations of H . Every component in this decomposition is continuous with respect to p and therefore factors through the C^* -subalgebra C_H generated in $C_p^*(G)$ by H . Thus, we get at most m irreducible representations of C_H ; by assumption they have dimension at most $n = \deg C_H$. Now $\dim \tau \leq mn$, what proves the statement.

III. It remains now to prove the theorem for the subgroup H , that is, a group representable as $H = ZK$, where the subgroup Z is central and K is compact and normal. Let $\pi : H \rightarrow \mathcal{B}(E)$ be a (norm-continuous) representation of H on a space E . Let $\rho = \pi|_K$, $\sigma = \pi|_Z$, and let C_π , C_ρ , C_σ be the C^* -algebras generated by π and by these restrictions.

Since ρ and σ commute, C_π is a continuous image of the maximal tensor product $C_\rho \otimes_{\max} C_\sigma$, so every irreducible representation of C_π generates an irreducible representation of $C_\rho \otimes_{\max} C_\sigma$. This tensor product may be described explicitly. As it is shown in theorem 2.10, C_σ is isomorphic to the algebra $C(M)$ for some compact set $M \subset \widehat{Z}$. By lemma 2.11, C_ρ is isomorphic to a finite direct sum of matrix algebras: $C_\rho = \oplus_{j=1}^m M_{n_j}(\mathbb{C})$. Thus,

$$C_\rho \otimes_{\max} C_\sigma = \bigoplus_{j=1}^m C(M) \otimes M_{n_j}(\mathbb{C})$$

(here \otimes is just the algebraic tensor product). It is easy to see that any irreducible representation of such an algebra has dimension at most $n_1^2 \cdots n_m^2$. \square

Corollary 7.6. *If G is a Moore group, then $\widehat{C}(G)$ is a strict pro- C^* -algebra.*

Proposition 7.7. *For a discrete group G which is not a Moore group, $\widehat{C}(G)$ is not a strict pro- C^* -algebra.*

Proof. According to theorem 5.1, $\widehat{C}(G)$ is the Banach algebra $C^*(G)$. By lemma 7.3, it is strict, so that the dimensions of all irreducible representations of G must be bounded by a common constant, and a fortiori, G must be a Moore group. \square

8. THE QUANTUM GROUP $SU_q(2)$

The quantum group $SU_q(2)$ was introduced by S. L. Woronowicz [40] as a deformation of the Lie group $SU(2)$. A detailed exposition of results concerning $SU_q(2)$ may be found in the book by Klimyk and Schmüdgen [21]. We will also stick to the notations of this book.

The main results in this section are Theorems 8.10 and 8.12. They state that the C^* -algebra A associated to the group $SU_q(2)$ is reflexive, i.e. $\widehat{\widehat{A}} = A$, and moreover, $SU_q(2)$ behaves as a usual compact group: the dual algebra \widehat{A} is the direct product of all finite-dimensional algebras generated by irreducible representations of A .

Definition 8.1. Let q be a real number, $q \neq 0, \pm 1$, and let $\mathcal{O}(SU_q(2))$ be the $*$ -Hopf algebra with generators a, c and relations

$$ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c, \quad a^*a + c^*c = 1, \quad aa^* + q^2c^*c = 1.$$

The counit ε is defined by $\varepsilon(a) = 1$, $\varepsilon(c) = 0$, and the comultiplication Δ and the antipode σ by

$$\begin{aligned}\Delta(a) &= a \otimes a - qc^* \otimes c, & \Delta(c) &= c \otimes a + a^* \otimes c, \\ \sigma(a) &= a^*, & \sigma(c) &= -qc, \\ \sigma(a^*) &= a, & \sigma(c^*) &= -q^{-1}c^*.\end{aligned}\tag{8}$$

(The last pair of equalities may be derived from the previous one). If we consider $\mathcal{O}(SU_q(2))$ with the strongest locally convex topology, we can define its C^* -envelope. It is a C^* -algebra which is called the C^* -algebra of the quantum group $SU_q(2)$. In this section we denote $\mathcal{A} = \mathcal{O}(SU_q(2))$ and its C^* -envelope just A .

In the original paper of Woronowicz [40] A itself was called the quantum group $SU_q(2)$; Klimyk and Schmüdgen [21, 4.1.4, 4.3.4] call $\mathcal{O}(SU_q(2))$ the algebra of coefficients of the group $SU_q(2)$ and A the C^* -algebra of $SU_q(2)$. The latter terminology is more standard now.

Corepresentations of A as a coalgebra are said to be representations of the quantum group $SU_q(2)$. It is custom to consider only finite-dimensional representations to avoid complications with topological tensor products. In this case it is clear that a corepresentation of A is the same as a representation of the conjugate algebra A^* . But for A^* one can naturally consider also infinite-dimensional representations, therefore we prefer to deal with this algebra. Multiplication on A^* will be denoted by $*$.

Let X be a finite-dimensional linear space with the basis $\{e_j : j = 1 \dots n\}$. By usual definition, a corepresentation $U : X \rightarrow X \otimes A$, $e_j \rightarrow \sum e_k \otimes u_{kj}$, is called unitary if

$$\sum_{k=1}^n u_{ki}^* u_{kj} = \sum_{k=1}^n u_{ik} u_{jk}^* = \delta_{ij}$$

for all i, j . This is equivalent to the set of identities $\sigma(u_{ij}) = u_{ji}^*$ [21, Proposition 11.11]. The last requirement has sense in the infinite-dimensional case as well, and below we show that this identity distinguishes involutive representations of A^* .

Let S be a continuous linear map from A^* into the space $B(H)$ of bounded operators on a Hilbert space H with the scalar product (\cdot, \cdot) . For any $x, y \in H$, there is a linear continuous functional

$$s_{xy}(\mu) = (S(\mu)y, x)\tag{10}$$

on A^* , which by duality belongs to A .

Lemma 8.2. *Let S be a continuous representation of A^* on a Hilbert space H . Then S is involutive if and only if for any $x, y \in H$ holds, in notations (10), $s_{xy}^* = \sigma(s_{yx})$.*

Proof. By definition of involution on A^* [7, p. 117], for any $\mu \in A^*$

$$\mu^*(a) = \overline{\mu((\sigma a)^*)}.\tag{11}$$

or $\mu((\sigma a)^*) = \overline{\mu^*(a)}$. Using this equality, we have:

$$\begin{aligned}\sigma(s_{yx})^*(\mu) &= \mu(\sigma(s_{yx})^*) = \overline{\mu^*(s_{yx})} = \overline{s_{yx}(\mu^*)} = \overline{(S(\mu^*)x, y)} = \\ &= (y, S(\mu^*)x) = (y, S(\mu)^*x) = (S(\mu)y, x) = s_{xy}(\mu),\end{aligned}$$

what proves the lemma. \square

Below we will consider only involutive representations of A^* .

Lemma 8.3. *Let S be a $*$ -representation of the algebra A^* on a Hilbert space H with an orthonormal basis $\{e_\alpha\}$. Then, in notations (8.4), for every $u \in A^* \otimes A^*$*

$$\Delta(s_{xy})(u) = \sum_{\alpha} (s_{xe_\alpha} \otimes s_{e_\alpha y})(u).$$

Proof. Recall that Δ acts from A to the maximal (in this case equal to minimal) tensor product $A \otimes A$. The conjugate space to $A \otimes A$ is by definition $A^* \otimes A^*$. For any $\mu, \nu \in A^*$ we have a direct computation, with all series absolutely convergent:

$$\begin{aligned} \Delta(s_{xy})(\mu \otimes \nu) &= s_{xy}(\mu * \nu) = (S(\mu * \nu)y, x) = (S(\mu)S(\nu)y, x) = \\ &= (S(\nu)y, S(\mu)^*x) = \sum_{\alpha} (S(\nu)y, e_\alpha)(e_\alpha, S(\mu)^*x) = \\ &= \sum_{\alpha} (S(\nu)y, e_\alpha)(S(\mu)e_\alpha, x) = \sum_{\alpha} s_{e_\alpha y}(\nu)s_{xe_\alpha}(\mu) = \\ &= \left(\sum_{\alpha} s_{xe_\alpha} \otimes s_{e_\alpha y} \right)(\mu \otimes \nu). \end{aligned}$$

Since A is nuclear, by lemma 6.11 the linear span of such tensors is dense in $A^* \otimes A^*$, what yields the equality needed. \square

Notations 8.4. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. If $h \in A^*$ is the Haar state of A , we have a scalar product $\langle a, b \rangle = h(ab^*)$ for $a, b \in A$. It is known that \mathcal{A} has a series of irreducible $(2l+1)$ -dimensional corepresentations T^l , $l \in \mathbb{N}_0$; denote t_{ij}^l , $i, j = -l, \dots, l$, the coefficients of T^l . Put $\tilde{t}_{ij}^l = t_{ij}^l / \langle t_{ij}^l, t_{ij}^l \rangle$, so that $\langle t_{ij}^l, \tilde{t}_{ij}^l \rangle = 1$. When this does not introduce any ambiguity, we omit the index l . It is known that t_{ij}^l are orthogonal with respect to $\langle \cdot, \cdot \rangle$ and \mathcal{A} is the linear span of t_{ij}^l . Note that this linear span is dense in A .

We need three technical lemmas treating the coefficients t_{ij}^l explicitly.

Lemma 8.5. *For $a \in A$, denote $\omega_a \in A^*$ the functional $\omega_a(x) = h(xa^*) = \langle x, a \rangle$ on A . Then $\omega_{t_{\alpha\beta}^l} * \omega_{\tilde{t}_{\gamma\delta}^m} = \delta_{\beta\gamma} \omega_{t_{\alpha\delta}^l}$ if $l = m$ and 0 otherwise.*

Proof. Since \mathcal{A} is dense in A , a continuous functional is defined by its values on $t_{\xi\eta}^k$. Using the formula for $\Delta(t_{ij}^l)$, we see that

$$\begin{aligned} (\omega_{t_{\alpha\beta}^l} * \omega_{\tilde{t}_{\gamma\delta}^m})(t_{\xi\eta}^k) &= (\omega_{t_{\alpha\beta}^l} \otimes \omega_{\tilde{t}_{\gamma\delta}^m})(\Delta t_{\xi\eta}^k) = (\omega_{t_{\alpha\beta}^l} \otimes \omega_{\tilde{t}_{\gamma\delta}^m})\left(\sum_{\zeta} t_{\xi\zeta}^k \otimes t_{\zeta\eta}^k\right) = \\ &= \sum_{\zeta} \omega_{t_{\alpha\beta}^l}(t_{\xi\zeta}^k) \omega_{\tilde{t}_{\gamma\delta}^m}(t_{\zeta\eta}^k) = \sum_{\zeta} \langle t_{\xi\zeta}^k, \tilde{t}_{\alpha\beta}^l \rangle \langle t_{\zeta\eta}^k, \tilde{t}_{\gamma\delta}^m \rangle = \\ &= \sum_{\zeta} \delta_{kl} \delta_{km} \delta_{\xi\alpha} \delta_{\zeta\beta} \langle t_{\alpha\beta}^l, \tilde{t}_{\alpha\beta}^l \rangle \langle t_{\beta\eta}^k, \tilde{t}_{\gamma\delta}^m \rangle = \\ &= \delta_{kl} \delta_{km} \delta_{\xi\alpha} \delta_{\beta\gamma} \delta_{\eta\delta} \langle t_{\gamma\delta}^m, \tilde{t}_{\gamma\delta}^m \rangle = \\ &= \delta_{kl} \delta_{km} \delta_{\xi\alpha} \delta_{\beta\gamma} \delta_{\eta\delta}. \end{aligned}$$

From the other side,

$$\delta_{\beta\gamma} \omega_{t_{\alpha\delta}^l}(t_{\xi\eta}^k) = \delta_{\beta\gamma} \langle t_{\xi\eta}^k, \tilde{t}_{\alpha\delta}^l \rangle = \delta_{\beta\gamma} \delta_{kl} \delta_{\xi\alpha} \delta_{\eta\delta}.$$

This proves the lemma. \square

Next lemma will not be used until in Theorem 8.10, but it introduces many notations needed earlier.

Lemma 8.6. *Let t_{ij}^l , $2l \in \mathbb{N}_0$, $i, j = 1 \dots \dim T^l$ be the coefficients of the l -th irreducible representation T^l of $SU_q(2)$. Denote $\tau_l = t_{-l+1, -l+1}^l$, then $\|\tau_l\| \rightarrow \infty$, $l \rightarrow \infty$.*

Proof. Let us first introduce the following notations [21, section 2.1]: for $n \in \mathbb{N}$

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a; q)_0 = 1.$$

This allows to define q -binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}$$

and the q -hypergeometric function ${}_2\varphi_1$:

$${}_2\varphi_1(a, b, c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n} \frac{z^n}{(q; q)_n}.$$

There is an explicit formula for the coefficients t_{ij}^l [21, p. 109]. If $i = j \leq 0$ (the only case we need), then

$$t_{ij}^l = q^{-(l+j)(j-i)} a^{-i-j} c^{i-j} \begin{bmatrix} l+i \\ i-j \end{bmatrix}_{q^{-2}} \begin{bmatrix} l-j \\ i-j \end{bmatrix}_{q^{-2}} {}_2\Phi_1(-l-j, l-j+1; i-j+1; q^{-2}, q^{-2}\zeta), \quad (12)$$

where ${}_2\Phi_1$ is defined via ${}_2\varphi_1$ as follows [21, p. 43]:

$${}_2\Phi_1(\alpha, \beta, \gamma; q, z) = {}_2\varphi_1(q^\alpha, q^\beta, q^\gamma; q, z),$$

so that

$${}_2\Phi_1(\alpha, \beta, \gamma; q^{-2}, q^{-2}\zeta) = {}_2\varphi_1(q^{-2\alpha}, q^{-2\beta}, q^{-2\gamma}; q^{-2}, q^{-2}\zeta),$$

Setting $i = j = -l + 1$, we get:

$$\begin{aligned} \tau_l &= a^{-2l+2} \begin{bmatrix} 2l-1 \\ 0 \end{bmatrix}_{q^{-2}}^2 {}_2\Phi_1(-1, 2l; 1; q^{-2}, q^{-2}\zeta) = \\ &= a^{-2l+2} {}_2\varphi_1(q^2, q^{-4l}; q^{-2}; q^{-2}, q^{-2}\zeta). \end{aligned}$$

Here ${}_2\varphi_1$ reduces to two summands:

$$\begin{aligned} {}_2\varphi_1(q^2, q^{-4l}; q^{-4l+2}; q^{-2}, q^{-2}\zeta) &= \sum_{n=0}^{\infty} \frac{(q^2; q^{-2})_n (q^{-4l}; q^{-2})_n (q^{-2}\zeta)^n}{(q^{-2}; q^{-2})_n (q^{-2}; q^{-2})_n} = \\ &= 1 + \frac{(1 - q^2)(1 - q^{-4l})q^{-2}\zeta}{(1 - q^{-2})^2} = \\ &= 1 + \frac{1 - q^{-4l}}{q^{-2} - 1} \zeta. \end{aligned}$$

Now, recalling that $\zeta = -qbc = q^2 c^* c$, we have

$$\tau_l = a^{-2l+2} \left(1 - \frac{q^{-4l} - 1}{q^{-2} - 1} q^2 c^* c \right).$$

As usual in a C^* -algebra, $\|\tau_l\| = \sup \|\pi(\tau_l)\|$ over all irreducible representations π of A . All these representations are described as follows [21, Corollary 4.3.20]. For every

$u \in \mathbb{T}$, there is a one-dimensional representation π_u^1 and a representation π_u^∞ on $\ell_2(\mathbb{N}_0)$ defined by

$$\begin{aligned} \pi_u^1(a) &= u, & \pi_u^1(c) &= 0; \\ \pi_u^\infty(a)e_n &= (1 - q^{2n})^{1/2}e_{n-1}, & \pi_u^\infty(c)e_n &= q^n u e_n. \end{aligned} \quad (13)$$

We see that $\|\pi_u^1(\tau_l)\| = |u|^{-2l+2} = 1$. In the infinite-dimensional case, we have $\pi_u^\infty(c)e_n = q^n u e_n$, $\pi_u^\infty(c^*)e_n = q^n \bar{u} e_n$, and $\pi_u^\infty(a)^{-1}e_n = (1 - q^{2n+2})^{-1/2}e_{n+1}$ so that

$$\begin{aligned} \pi_u^\infty(\tau_l)e_n &= a^{-2l+2} \left(\left(1 - \frac{q^{-4l} - 1}{q^{-2} - 1} q^2 \right) q^{2n} |u|^2 e_n \right) = \\ &= \left(q^{2n} - \frac{q^{-4l} - 1}{q^{-2} - 1} q^{2n+2} \right) \left((1 - q^{2n+2}) \cdots (1 - q^{2(n+2l-2)}) \right)^{-1/2} e_{n+2l-2} \end{aligned}$$

For $n = 0$ this gives

$$\pi_u^\infty(\tau_l)e_0 = \left(1 - \frac{q^{-4l} - 1}{q^{-2} - 1} q^2 \right) \left((1 - q^2) \cdots (1 - q^{4l-4}) \right)^{-1/2} e_{2l-2} \equiv \alpha_l e_{2l-2}.$$

The infinite product $\left((1 - q^2) \cdots (1 - q^{4l-4}) \right)^{-1/2}$ converges to a limit β , so we have $\alpha_l \sim -\beta q^{-4l+2} \rightarrow \infty$, $l \rightarrow \infty$, and thus $\|\tau_l\| \rightarrow \infty$, what was to prove. \square

Lemma 8.7. *For coefficients of an irreducible representation T^l , we have $\sigma(\tilde{t}_{ij}^*) = \tilde{t}_{ji}$.*

Proof. We will show first that $\sigma^2(t_{ij}) = q^{2(i-j)}t_{ij}$. From the formula (12) we see that $t_{ij} = a^{-i-j}c^{i-j}P_{ij}(\zeta)$ with some polynomial P_{ij} , where $\zeta = -qbc = q^2c^*c$. Since σ is an anti-homomorphism, σ^2 is a homomorphism. Then we have:

$$\begin{aligned} \sigma^2(c) &= -q(-qc) = q^2c \\ \sigma^2(c^*) &= \sigma(-q^{-1}c^*) = -q^{-1}(-q^{-1}c^*) = q^{-2}c^* \\ \sigma^2(c^*c) &= \sigma^2(c^*)\sigma^2(c) = q^{-2}c^*q^2c = c^*c \\ \sigma^2(P_{ij}(\zeta)) &= P_{ij}(\zeta)\sigma^2(a) = \sigma(a^*) = a \\ \sigma^2(t_{ij}) &= [\sigma^2(a)]^{-i-j}[\sigma^2(c)]^{i-j}\sigma^2(P_{ij}(\zeta)) = a^{-i-j}[q^2c]^{i-j}P_{ij}(\zeta) = q^{2(i-j)}t_{ij}. \end{aligned}$$

Now, by lemma 8.2 $t_{ij}^* = \sigma t_{ji}$. Then $\sigma(t_{ij}^*) = \sigma^2(t_{ji}) = q^{2(j-i)}t_{ji}$.

By [21, Theorem 4.17], we have $\langle t_{ij}^l, t_{ij}^l \rangle = q^{2j}[2l+1]_q^{-1}$, $\langle t_{ji}^l, t_{ji}^l \rangle = q^{2i}[2l+1]_q^{-1}$, so that $\langle t_{ji}^l, t_{ji}^l \rangle / \langle t_{ij}^l, t_{ij}^l \rangle = q^{2i}/q^{2j} = q^{2(i-j)}$.

For the normalized coefficients now holds

$$\sigma((\tilde{t}_{ij}^*)^*) = \frac{\sigma((t_{ij}^l)^*)}{\langle t_{ij}^l, t_{ij}^l \rangle} = \frac{q^{2(j-i)}t_{ji}^l}{\langle t_{ij}^l, t_{ij}^l \rangle} = q^{2(j-i)}\tilde{t}_{ji}^l \frac{\langle t_{ji}^l, t_{ji}^l \rangle}{\langle t_{ij}^l, t_{ij}^l \rangle} = q^{2(j-i)}\tilde{t}_{ji}^l \cdot q^{2(i-j)} = \tilde{t}_{ji}^l.$$

\square

Lemma 8.8. *Let S be a continuous linear map from A^* into the space $B(H)$ of bounded operators on a Hilbert space H with the scalar product (\cdot, \cdot) . Then for every $a \in A$ the equality*

$$(Px, y) = \langle (Sx, y), a \rangle \quad (14)$$

defines a bounded linear operator P on H .

Proof. Note first that (Sx, y) is a continuous linear functional on A^* for any $x, y \in H$, and by duality it belongs to A , so that the right-hand side of (14) is well defined. Obviously P is linear, so we need to prove its continuity only. By assumption $\|S(\mu)\| \leq p(\mu)$, $\mu \in A^*$, for a continuous seminorm p on A^* . By definition of the topology of A^* there is a totally bounded, in particular bounded set $K \subset A$ such that

$$p(\mu) \leq \sup_{\xi \in K} |\mu(\xi)|.$$

Clearly K is contained in a multiple of the unit ball B in A : $K \subset \lambda B$ with some $\lambda > 0$. Then

$$p(\mu) \leq \sup_{\xi \in \lambda B} |\mu(\xi)| = \lambda \sup_{\xi \in B} |\mu(\xi)|.$$

Let $B^\circ = \{\mu \in A^* : |\mu|_B| \leq 1\}$ be the polar of B , then the previous formula shows that $p(\mu) \leq \lambda$ for all $\mu \in B^\circ$. Now we can estimate the norm of (Sx, y) as an element of A :

$$\begin{aligned} \|(Sx, y)\| &= \sup_{\mu \in B^\circ} |(Sx, y)(\mu)| = \sup_{\mu \in B^\circ} |(S(\mu)x, y)| \leq \sup_{\mu \in B^\circ} \|x\| \cdot \|y\| \cdot \|S(\mu)\| \leq \\ &\leq \|x\| \cdot \|y\| \cdot \sup_{\mu \in B^\circ} p(\mu) \leq \|x\| \cdot \|y\| \cdot \lambda. \end{aligned}$$

And then we have

$$|(Px, y)| = \langle (Sx, y), a \rangle = h((Sx, y)a^*) \leq \|(Sx, y)a^*\| \leq \|(Sx, y)\| \cdot \|a\| \leq \lambda \|x\| \cdot \|y\| \cdot \|a\|,$$

what shows that P is indeed a bounded operator. \square

Next we come to an essential statement: a direct sum decomposition of a representation of $SU_q(2)$. It is not though a final description: Theorem 8.10 states that in such a decomposition only finite number of different irreducible representation is involved. As a byproduct we get that A^* has no irreducible representations except T^l .

Lemma 8.9. *Every continuous representation of A^* is equal to the direct sum of its irreducible (finite-dimensional) representations, each of them equivalent to T^l for some l .*

Proof. We will show that the classical proof [28, Chapter IV, §2] can be adapted to our situation. Let S be a continuous representation of A^* in a Hilbert space H and let e_α be its orthonormal basis and (\cdot, \cdot) the scalar product in H . Then the coefficients $s_{xy}(\mu) = (S(\mu)y, x)$ are linear continuous functionals on A^* , and by duality they belong to A . We introduce a family of operators P_{ij}^l , $2l \in \mathbb{N}_0$, $i, j = -l, \dots, l$, by

$$(P_{ij}^l x, y) = \langle (Sx, y), \tilde{t}_{ij}^l \rangle$$

(by lemma 8.8 they are continuous). These operators have the following properties:

- (i) $P_{ij}^{l*} = P_{ji}^l$
- (ii) $P_{ij}^l P_{kn}^m = \delta_{lm} \delta_{jk} P_{in}^l$

Denote $s_{xy} = (Sy, x)$.

(i) We will calculate in general the adjoint of the operator P_a defined by (14).

$$\begin{aligned} (P_a^* x, y) &= (x, P_a y) = \overline{(P_a y, x)} = \overline{\langle (S y, x), a \rangle} = \overline{\langle s_{xy}, a \rangle} = \overline{h(s_{xy} a^*)} = \overline{h(\sigma(s_{xy} a^*))} = \\ &= \overline{h(\sigma(a^*) \sigma(s_{xy}))} = \overline{h(\sigma(a^*) s_{yx}^*)} = h((\sigma(a^*) s_{yx}^*)^*) = \\ &= h(s_{yx} (\sigma(a^*))^*) = \langle s_{yx}, \sigma(a^*) \rangle = (P_{\sigma(a^*)} x, y). \end{aligned}$$

Thus, $P_a^* = P_{\sigma(a^*)}$. In our case $a = \tilde{t}_{ij}^l$, and by lemma 8.7 $\sigma((\tilde{t}_{ij}^l)^*) = \tilde{t}_{ji}^l$. Thus, $(P_{ij}^l)^* = P_{ji}^l$.

(ii) Using lemma 8.3, we have:

$$\begin{aligned} (P_{ij}^l P_{kn}^m x, y) &= (P_{kn}^m x, P_{ij}^{l*} y) = \sum_r (P_{kn}^m x, e_r) (e_r, P_{ij}^{l*} y) = \sum_r (P_{kn}^m x, e_r) (P_{ij}^l e_r, y) = \\ &= \sum_r \langle (S x, e_r), \tilde{t}_{kn}^m \rangle \langle (S e_r, y), \tilde{t}_{ij}^l \rangle = \sum_r (s_{y e_r} \otimes s_{e_r x}) (\omega_{\tilde{t}_{ij}^l} \otimes \omega_{\tilde{t}_{kn}^m}) = \\ &= \Delta(s_{yx}) (\omega_{\tilde{t}_{ij}^l} \otimes \omega_{\tilde{t}_{kn}^m}) = s_{yx} (\omega_{\tilde{t}_{ij}^l} * \omega_{\tilde{t}_{kn}^m}) = s_{yx} (\delta_{lm} \delta_{jk} \omega_{\tilde{t}_{in}^l}) = \\ &= \delta_{lm} \delta_{jk} \langle s_{yx}, \tilde{t}_{in}^l \rangle = \delta_{lm} \delta_{jk} (P_{in}^l x, y). \end{aligned}$$

From (i) and (ii) we see that $P_{ii}^{l*} = P_{ii}^l$ and $(P_{ii}^l)^2 = P_{ii}^l$, that is, P_{ii}^l is an orthogonal projector. Moreover, from (ii) it follows that P_{ii}^l are orthogonal to each other.

Next we prove that $H = \oplus_{i,l} P_{ii}^l H$. Suppose the opposite, i.e. that $x \in H$ is orthogonal to $P_{ii}^l H$ for all i and l . From (ii) we have $P_{ii}^l P_{ij}^l = P_{ij}^l$, so that image of P_{ij}^l is contained into $P_{ii}^l H$. Then x is orthogonal to $P_{ij}^l H$ for all i, j, l . In particular, $(P_{ij}^l x, x) = \langle (S x, x), \tilde{t}_{ij}^l \rangle = 0$. Since \tilde{t}_{ij}^l form an orthogonal basis in A , it follows that $(S x, x) = 0$. In particular, its value on the unit e of A^* is 0, i.e. $(x, x) = (S x, x)(e) = 0$, whence $x = 0$.

Let $P^l = \sum_{i=-l}^l P_{ii}^l$ and $M^l = P^l H$. We show that every M^l is invariant under S , and more precisely, that $S P_{ii}^l = \sum_{j=-l}^l t_{ji}^l P_{ji}^l$. For any $x, y \in H$ we have in the r.h.s.:

$$\begin{aligned} \langle \sum_{j=-l}^l t_{ji}^l (P_{ji}^l x, y), \tilde{t}_{nk}^m \rangle &= \sum_{j=-l}^l (P_{ji}^l x, y) \langle t_{ji}^l, \tilde{t}_{nk}^m \rangle = \sum_{j=-l}^l \langle (S x, y), \tilde{t}_{ji}^l \rangle \delta_{lm} \delta_{jn} \delta_{ik} = \\ &= \delta_{lm} \delta_{ik} \langle (S x, y), \tilde{t}_{ni}^l \rangle = \delta_{lm} \delta_{ik} \langle (S x, y), \tilde{t}_{nk}^m \rangle. \end{aligned}$$

In the l.h.s.

$$\begin{aligned} (S P_{ii}^l x, y) &= (P_{ii}^l x, S^* y) = \sum_{\alpha} (P_{ii}^l x, e_{\alpha}) (e_{\alpha}, S^* y) = \sum_{\alpha} \langle (S x, e_{\alpha}), \tilde{t}_{ii}^l \rangle (S e_{\alpha}, y) = \\ &= \sum_{\alpha} \langle s_{e_{\alpha} x}, \tilde{t}_{ii}^l \rangle s_{y e_{\alpha}}. \end{aligned}$$

For any m, n, k now

$$\begin{aligned} \langle (S P_{ii}^l x, y), \tilde{t}_{nk}^m \rangle &= \sum_{\alpha} \langle s_{y e_{\alpha}}, \tilde{t}_{nk}^m \rangle \langle s_{e_{\alpha} x}, \tilde{t}_{ii}^l \rangle = \sum_{\alpha} (s_{y e_{\alpha}} \otimes s_{e_{\alpha} x}) (\omega_{\tilde{t}_{nk}^m} \otimes \omega_{\tilde{t}_{ii}^l}) = \\ &= \Delta(s_{yx}) (\omega_{\tilde{t}_{nk}^m} \otimes \omega_{\tilde{t}_{ii}^l}) = s_{yx} (\omega_{\tilde{t}_{nk}^m} * \omega_{\tilde{t}_{ii}^l}) = \\ &= \delta_{lm} \delta_{ik} s_{yx} (\omega_{\tilde{t}_{ni}^l}) = \delta_{lm} \delta_{ik} \langle s_{yx}, \tilde{t}_{nk}^m \rangle \end{aligned}$$

Due to the density of the linear span of \tilde{t}_{nk}^m in A , this proves our equality.

Denote $H_i = P_{ii}^l H$ and $M^l = \bigoplus_{i=-l}^l H_i$. It remains to show that $S|_{M^l}$, if $M^l \neq 0$, is equivalent to a multiple of T^l . Till the end of the proof we fix an index l and omit it in the notations. Choose an orthonormal basis $e^r \in H_0$ and define $e_i^r = P_{i0} e^r$ (in particular, $e_0^r = e^r$). Then $e_i^r \in H_i$, so $(e_i^r, e_k^s) = 0$ if $i \neq k$; for $i = k$ we have $(e_i^r, e_i^s) = (P_{i0} e^r, P_{i0} e^s) = (P_{i0}^* P_{i0} e^r, e^s) = (P_{0i} P_{i0} e^r, e^s) = (P_{00} e^r, e^s) = \delta_{rs}$. For fixed i they span $P_{i0} H_0$; due to (ii), $P_{i0} H = P_{i0} H_0$ so that e_i^r span H_i . Thus, e_i^r form an orthonormal basis in H_i .

It remains to show that S is equivalent to T^l on the linear span of e_i^r for fixed r . For this, we need the equality $(S e_i^r, e_k^r) = t_{ki}$, and it can be proved by direct calculation:

$$\begin{aligned} \langle (S e_i^r, e_k^r), \tilde{t}_{\alpha\beta}^m \rangle &= (P_{\alpha\beta}^m e_i^r, e_k^r) = \delta_{lm} (P_{k0}^* P_{\alpha\beta} P_{i0} e^r, e^r) = \delta_{lm} (P_{0k} P_{\alpha\beta} P_{i0} e^r, e^r) = \\ &= \delta_{lm} \delta_{k\alpha} (P_{0\beta} P_{i0} e^r, e^r) = \delta_{lm} \delta_{k\alpha} \delta_{i\beta} (P_{00} e^r, e^r) = \delta_{lm} \delta_{k\alpha} \delta_{i\beta}, \end{aligned}$$

while $\langle t_{ki}, \tilde{t}_{\alpha\beta}^m \rangle = \delta_{lm} \delta_{k\alpha} \delta_{i\beta}$. \square

Theorem 8.10. *Every continuous representation of A^* is equivalent to a finite sum of its irreducible representations.*

Proof. Let π be a continuous representation of A^* and let p be its norm. Let $\pi = \bigoplus_{\alpha \in \mathcal{J}} \pi_\alpha$ be the decomposition of π according to lemma 8.9 and let $l(\alpha)$ be such that π_α is equivalent to $T^{l(\alpha)}$. The coefficients of all π_α , $\alpha \in \mathcal{J}$, are continuous with respect to p . Every set in A , equicontinuous on A^* , is totally bounded, in particular, bounded (by norm). By lemma 8.6, $\|t_{-l+1, -l+1}^l\| \rightarrow \infty$, $l \rightarrow \infty$. Thus, the set $\{l(\alpha) : \alpha \in \mathcal{J}\}$ must be finite. \square

Lemma 8.11. *The algebra $\hat{A} = A^{*\diamond}$ is a Hopf pro- C^* -algebra.*

Proof. The counit of A^* is a $*$ -homomorphism and is extended to a morphism of \hat{A} by universality property. It remains to check that the comultiplication Δ_* and antipode σ^* of A^* may be extended by continuity to \hat{A} , i.e. that they are continuous with respect to any C^* -seminorm p on A^* . Due to theorem 8.10, it is sufficient to consider only seminorms p_l of the irreducible representations T^l .

As in a general Banach algebra, multiplication of A acts from the projective tensor product $A \hat{\otimes} A$ to A (and not from $A \otimes A$, since A is not strict). Thus, Δ_* maps A^* to $(A \hat{\otimes} A)^*$. Since every T^l is finite-dimensional, the inclusion $i : A^* \otimes A^* \rightarrow \hat{A} \otimes \hat{A}$ may be extended to $(A \hat{\otimes} A)^*$ by continuity. Then its composition with Δ_* gives a continuous homomorphism $\delta : A^* \mapsto \hat{A} \otimes_{\max} \hat{A}$, and when extended by continuity to \hat{A} , it gives the comultiplication $\hat{\Delta}$ on \hat{A} .

The antipode requires a bit more effort. By lemma 8.2, $\sigma(t_{ij}^l) = (t_{ij}^l)^*$, and from explicit formulas [21, (40)-(44)] it may be seen that $(t_{ij}^l)^* = (-q)^{j-i} t_{-i, -j}^l$. So, for q with $|q| > 1$

$$p_l(\sigma^* \mu) \leq C_l \max_{i,j} |\mu(\sigma t_{ij}^l)| = C_l \max_{i,j} |\mu((t_{ij}^l)^*)| = C_l q^{2l} \max_{i,j} |\mu(t_{ij}^l)| \leq C_2 p_l(\mu).$$

This proves the lemma. \square

Since \hat{A} is the direct product of finite-dimensional spaces, it is stereotype (lemma 5.11). So we can define its second dual $\hat{\hat{A}}$, and the following theorem proves that

$\widehat{\widehat{A}} = A$. Thus, A belongs to the category \mathfrak{SM} of reflexive Hopf pro- C^* -algebras defined in section 6.

Theorem 8.12. *The algebra A is reflexive with respect to duality functor introduced in section 6: $\widehat{\widehat{A}} = A$.*

Proof. By lemma 8.10, every representation of A^* is equivalent to a finite sum of irreducible representations; then (cf. theorem 2.12) \widehat{A} is the direct product of the (finite-dimensional) algebras C_π generated by all irreducible representations π . Next, $(\widehat{A})^*$ is the direct sum of C_π^* , the algebras of coefficients of these representations; by [21, Theorem 4.17] this is exactly \mathcal{A} . Finally, the C^* -envelope $\mathcal{A}^\diamond = \widehat{\widehat{A}}$ of \mathcal{A} is by definition A . \square

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